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Centers of quasi-homogeneous polynomial differential equations of degree three

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ABSTRACT

We characterize the centers of the quasi-homogeneous planar polynomial differential systems of degree three. Such systems do not admit isochronous centers. At most one limit cycle can bifurcate from the periodic orbits of a center of a cubic homogeneous polynomial system using the averaging theory of first order.

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1. Introduction and statement of the results

Poincaré in [24] was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a center is a singular point surrounded by a neighborhood filled of closed orbits with the unique exception of the singular point.

Since then the center–focus problem, i.e. the problem to distinguish when a singular point is either a focus or a center is one of the hardest problems in the qualitative theory of planar differential systems, see for instance [1] and the references quoted there. This paper deals mainly with the characterization of the center problem for the class of quasi-homogeneous polynomial differential systems of degree 3.

In the literature we found classifications of polynomial differential systems having a center. For the quadratic systems we refer to the works of Dulac [8], Kapteyn [12,13], Bautin [3] among others. In [27] Schlomiuk, Guckenheimer and Rand gave a brief history of the center problem for quadratic systems.

There are many partial results about centers for polynomial differential systems of degree greater than two. Some of them (closed to our work) are for instance, the classification by Malkin [18] and Vulpe and Sibirskii [29] about the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree three. Note that for polynomial differential systems of the form linear with homogeneous nonlinearities of degree $k > 3$ the centers are not classified. However, there are some results for $k = 4, 5$, see for instance the works by Chavarriga and Giné [5,6]. It seems difficult for the moment to obtain a complete classification of the centers for the class of all polynomial differential systems of degree 3. Actually, there are some subclasses of cubic systems well studied like the ones of Rousseau and Schlomiuk [25] and the ones of Żołądek [30,31]. Some centers for arbitrary degree polynomial differential systems have been studied in [17].

In what follows we denote by $\mathbb{R}[x, y]$ the ring of all polynomials in the variables x and y and coefficients in the real numbers $\mathbb{R}$. In this work we consider polynomial differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

with $P, Q \in \mathbb{R}[x, y]$ and its corresponding vector field $\mathcal{X} = (P, Q)$. Here the dot denotes derivative with respect to the time t (independent variable). The *degree* of the differential polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

System (1) is a *quasi-homogeneous polynomial differential system* if there exist natural numbers s_1, s_2, d such that for an arbitrary non-negative real number α it holds

$$P\left(\alpha^{s_1} x, \alpha^{s_2} y\right) = \alpha^{s_1+d-1} P(x, y), \qquad Q\left(\alpha^{s_1} x, \alpha^{s_2} y\right) = \alpha^{s_2+d-1} Q(x, y). \tag{2}$$

The natural numbers s_1 and s_2 are the *weight exponents* of system (1) and d is the *weight degree* with respect to the weight exponents s_1 and s_2. When $s_1 = s_2 = s$ we obtain the classical *homogeneous polynomial differential system* of degree $s + d - 1$.