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Normal bundles of convex bodies

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A B S T R A C T

The normal bundle of an (o-symmetric, smooth and strictly) convex body C in $\mathbb{E}^d$ gives rise to a closed convex cone in $\mathbb{E}^{d^2}$, the *normal bundle cone* $\mathcal{N}_C$ of C. This article deals with properties of the cone $\mathcal{N}_C$ and relations between properties of $\mathcal{N}_C$ and C:
The family of normal bundle cones is a meager subset of the space of all closed convex cones in $\mathbb{E}^{d^2}$. To find out whether a cone is a normal bundle cone, a simple criterion is given. The dimension of a normal bundle cone $\mathcal{N}_C$ is between $\frac{1}{2}d(d+1)$ and d^2. The lower bound is attained precisely for ellipsoids. As the dimension of $\mathcal{N}_C$ increases, the ellipsoidal character of the convex body C and the family of linear vector fields which are tangent on the boundary of C decrease. For generic C the dimension of $\mathcal{N}_C$ is d^2. For $d = 2, 3$ a complete description of the situation is given. Next, symmetry properties are studied. The cone $\mathcal{N}_C$ coincides with its polar, its polar transpose, or its transpose, if and only if C is a disc with rotational symmetry of order 4, a Radon disc, a Euclidean ball, or an ellipsoid. A conjecture deals with isometries and linear automorphisms of $\mathcal{N}_C$ and C. A relation between symmetry properties of a pair of normal bundle cones and orthogonality in two normed spaces is stated. There is a bijection between the family of certain faces of $\mathcal{N}_C$ and the family of planar shadow boundaries of C with respect to parallel illumination. A conjecture deals with a more general case.

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1. Introduction

We represent the normal bundle of a (proper, o-symmetric, smooth and strictly) convex body C in $\mathbb{E}^d$ by a closed convex cone in $\mathbb{E}^{d^2}$, the *normal bundle cone* $\mathcal{N}_C$ of C. In the following we first specify properties of the cone $\mathcal{N}_C$ and then study relations between features of $\mathcal{N}_C$ and C.

The specified properties include a description of the exposed rays, a sub-additivity property, and an integral representation of the unit matrix. The normal bundle cones form a meager subset of the space of all closed convex cones in $\mathbb{E}^{d^2}$ − if the space is endowed with a natural topology which makes it Baire. A simple criterion is given to single out the normal bundle cones among the family of all convex cones (Section 2). The dimension of a normal bundle cone $\mathcal{N}_C$ is between $\frac{1}{2}d(d+1)$ and d^2. The lower bound is attained if and only if C is an ellipsoid. If d is fixed, the ellipsoidal character of C and the family of tangent linear vector fields on the boundary of C decrease as the dimension of $\mathcal{N}_C$ increases. For generic C the dimension of $\mathcal{N}_C$ is d^2. If $d = 2$ this dimension is 3 for ellipses and 4 for all other bodies; for $d = 3$ it attains the values 6 for ellipsoids, 8 for linear images of bodies of revolution, and 9 in all other cases. Dimension 7 does not occur (Section 3). Next, symmetry properties of normal bundle cones are investigated. The convex bodies C for which $\mathcal{N}_C$ coincides with its polar, its polar transpose, or its transpose are for $d = 2$ the Radon discs and discs with rotational symmetry of order 4 and for $d \geqslant 3$ the Euclidean balls and the ellipsoids. A conjecture − if true − implies that the groups of isometries and of linear automorphisms of the cone $\mathcal{N}_C$, and the convex body C are isomorphic. We then consider a more general notion of symmetry of orthogonality with respect to two normed spaces and show that it is equivalent to an inclusion between the normal bundle cones assigned to the unit balls of the norms (Section 4). There is a simple one-to-one correspondence between a special family of exposed faces of $\mathcal{N}_C$ and the family of planar shadow boundaries of C with respect to illumination parallel to lines. A conjecture claims that the family of all exposed faces of $\mathcal{N}_C$ and the family of all planar shadow boundaries of C with respect to illumination parallel to subspaces of any dimension correspond to each other (Section 5).

Since the normal bundle cone $\mathcal{N}_C$ of a convex body C determines the convex body up to dilatation, one could have expected relations between properties of $\mathcal{N}_C$ and C. The surprising fact is that there are relations between properties of $\mathcal{N}_C$ and C which a priori seemed totally unrelated. Simple properties of the cone $\mathcal{N}_C$ (dimension, symmetry, face structure) shed light on more sophisticated properties of the body C (tangent vector fields, ellipsoidal character, orthogonality, planar shadow boundaries).

While we are not aware that general normal bundle cones have been considered before, a special case, the cone $\mathcal{N}_{B^d}$ of positive semi-definite quadratic forms, plays a role in number theory, in particular in arithmetic number theory and in the geometry of numbers. In the latter it is present in the context of reduction of positive definite quadratic forms, of Voronoĭ's theory of lattice packing of balls and of Epstein's zeta function. Discrete versions of the integral formula in Theorem 1 appear in the extensions of Voronoĭ's