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# Asymptotic expansions for the constants of Landau and Lebesgue

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ABSTRACT

The constants of Landau and Lebesgue are defined, for all integers $n \geqslant 0$, in order, by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t,$$

which play important roles in the theories of complex analysis and Fourier series, respectively. Certain inequalities and asymptotic expansions for the constants $G_n$ and $L_n$ have been investigated by many authors. Here we aim at establishing new asymptotic expansions for the constants $G_n$ and $L_n$ of Landau and Lebesgue, respectively, by mainly using Bell polynomials and the partition function.



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## 1. Introduction and preliminaries

The Landau constants are defined by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \bigl(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{N} := \{1, 2, 3, \ldots\}\bigr), \tag{1.1}$$

which play an important role in the theory of complex analysis. More precisely, in 1913, Landau [19] proved that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function in the unit disc $\mathcal{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$, $\mathbb{C}$ being the set of complex numbers, which satisfies $|f(z)| < 1$ for all $z \in \mathcal{D}$, then $|\sum_{k=0}^{n} a_k| \leqslant G_n$ $(n \in \mathbb{N}_0)$ whose bounds are seen to be optimal.

The Lebesgue constants are defined by

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t \quad (n \in \mathbb{N}_0), \tag{1.2}$$

which play an important role in the theory of Fourier series. More in detail, in 1906, Lebesgue [20] proved the following result: Assume a function $f$ is integrable on the interval $[-\pi, \pi]$ and $S_n(f, x)$ is the $n$th partial sum of the Fourier series of $f$. That is,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\, \mathrm{d}t \quad (k \in \mathbb{N}_0) \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\, \mathrm{d}t \quad (k \in \mathbb{N})$$

and

$$S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n} \bigl(a_k \cos(kx) + b_k \sin(kx)\bigr) \quad (n \in \mathbb{N}_0),$$

where the empty sum is (as usual, throughout this paper) understood to be nil. If $|f(x)| \leqslant 1$ for all $x \in [-\pi, \pi]$, then

$$S_n(f, x) \leqslant L_n \quad (n \in \mathbb{N}_0). \tag{1.3}$$

It is noted that $L_n$ is the smallest possible constant for which the inequality (1.3) holds true for all continuous functions $f$ on $[-\pi, \pi]$.

Certain inequalities and asymptotic expansions for the constants $G_n$ and $L_n$ have been investigated by many authors (see, *e.g.*, [5,6,10,16,23–27,33]). Here, in this paper, we aim at establishing new asymptotic expansions for the constants $G_n$ and $L_n$ of Landau and Lebesgue, respectively, by mainly using Bell polynomials and the partition function.

For our purpose, the following lemmas are required in the sequel.