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Multi-tiling and Riesz bases

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Abstract

Let S be a bounded, Riemann measurable set in $\mathbb{R}^d$, and Λ be a lattice. By a theorem of Fuglede, if S tiles $\mathbb{R}^d$ with translation set Λ, then S has an orthogonal basis of exponentials. We show that, under the more general condition that S *multi-tiles* $\mathbb{R}^d$ with translation set Λ, S has a Riesz basis of exponentials. The proof is based on Meyer's quasicrystals.

Keywords: Riesz bases; Tiling; Quasicrystals

1. Introduction

1.1. Let S be a bounded, measurable set in $\mathbb{R}^d$. A sequence $\Lambda \subset \mathbb{R}^d$ is called a *spectrum* for S if the system of exponential functions

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \qquad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle},$$

is a complete orthogonal system in $L^2(S)$. The existence of a spectrum Λ for S provides a unique and stable expansion of any $f \in L^2(S)$ into a "non-harmonic" Fourier series with frequencies in Λ.

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The study of spectra was initiated by Fuglede [2] who suggested a connection with the concept of tiling. We say that S tiles $\mathbb{R}^d$ with translation set Λ if the sets $S + \lambda$ $(\lambda \in \Lambda)$ are disjoint and cover the whole space up to measure zero, that is,

$$\sum_{\lambda \in \Lambda} \mathbb{1}_S(x - \lambda) = 1 \quad \text{(a.e.)}.$$

Theorem. *(Fuglede [2].) Let Λ be a lattice. If S tiles $\mathbb{R}^d$ with translation set Λ, then the dual lattice Λ^* is a spectrum for S, and also the converse is true.*

By a lattice we mean the image of $\mathbb{Z}^d$ under some invertible linear transformation. The dual lattice Λ^* is the set of all vectors $\lambda^* \in \mathbb{R}^d$ such that $\langle \lambda, \lambda^* \rangle \in \mathbb{Z}$, $\lambda \in \Lambda$.

Fuglede conjectured that S admits a spectrum if and only if it can tile $\mathbb{R}^d$ with some translation set. This conjecture inspired extensive research, see the surveys [6,8] and the references therein for more information, including the present state of the conjecture.

1.2. Fuglede's conjecture suggests that the existence of a spectrum is a rather special property of the set S. Indeed, it is known that some very simple sets S do not admit an orthogonal basis of exponentials. For example, it is easy to construct a union of two intervals on $\mathbb{R}$ which does not admit such a basis (see also [12]). In dimension two, a convex polygon has a spectrum only if it is (up to an affine transformation) either a square or a hexagon [5]. The ball in any dimension $d \geqslant 2$ has no spectrum [2,4].

The exponential system $E(\Lambda)$ is said to be a *Riesz basis* for S if the mapping $f \mapsto \{\langle f, e_\lambda \rangle\}_{\lambda \in \Lambda}$ is bounded and invertible from $L^2(S)$ onto $\ell^2(\Lambda)$ (but not necessarily unitary, as in the case when Λ is a spectrum). This condition still allows a unique and stable expansion of any $f \in L^2(S)$ into a Fourier series with frequencies in Λ, see [20], but it is not as rigid as the orthogonality requirement for the exponentials (for example, it is stable under small perturbations of the frequencies). Thus, one might hope that sets with no spectrum should at least have a Riesz basis of exponentials.

This was indeed established in some particular cases. It is a recent result due to Kozma and Nitzan [10] that any finite union of intervals admits a Riesz basis of exponentials. In dimension two it was proved by Lyubarskii and Rashkovskii [14] that any convex, symmetric polygon has such a basis.

In spite of this progress, however, there are still relatively few results on the existence of a Riesz basis of exponentials. In particular, it is still an open problem if such a basis exists for the ball in dimension $d \geqslant 2$. To the best of our knowledge, the only result in the literature in dimension greater than two is that of Marzo [16], who showed that there exists a Riesz basis of exponentials for S whenever $S \subset \mathbb{R}^d$ is a finite union of disjoint translates of the unit cube.

1.3. In this paper we prove a result in spirit of Fuglede's theorem, but we consider the more general setting when S tiles $\mathbb{R}^d$ *with multiplicity*. The set S is said to k-tile $\mathbb{R}^d$ with the translation set Λ if almost every point gets covered exactly k times by the sets $S + \lambda$ $(\lambda \in \Lambda)$, that is,

$$\sum_{\lambda \in \Lambda} \mathbb{1}_S(x - \lambda) = k \quad \text{(a.e.)}. \tag{1}$$

Clearly, the family of sets which k-tile $\mathbb{R}^d$ for some k is much larger than the family of sets which just tile (i.e. 1-tile). We will prove: