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Invariant hypersurfaces of endomorphisms of projective varieties

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Abstract

We consider surjective endomorphisms f of degree > 1 on projective manifolds X of Picard number one and their f^{-1}-stable hypersurfaces V, and show that V is rationally chain connected. Also given is an optimal upper bound for the number of f^{-1}-stable prime divisors on (not necessarily smooth) projective varieties.

MSC: 14E20; 14J45; 37F10; 32H50

Keywords: Endomorphism; Projective spaces; Iteration; Complex dynamics

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. Theorems 1.1–1.3 below are our main results. We refer to [19, Definition 2.34] for the definitions of *Kawamata log terminal* (*klt*) and *log canonical singularities*. See S.-W. Zhang [32, §1.2, §4.1] for the dynamic Manin–Mumford conjecture solved for the pair (X, f) as in the conclusion part of Theorem 1.1 below, and [5] for a related result on endomorphisms of (not necessarily projective) compact complex manifolds.

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Theorem 1.1. *Let X be a normal projective variety of dimension $n \geqslant 2$, V_i $(1 \leqslant i \leqslant s)$ prime divisors, H an ample Cartier divisor, and $f : X \to X$ an endomorphism with* $\deg(f) = q^n > 1$ *such that* (*for all* i):

(1) *X has only log canonical singularities around $\bigcup V_i$;*
(2) *V_i is Cartier and $V_i \equiv d_i H$ (numerically) for some $d_i > 0$; and*
(3) *$f^{-1}(V_i) = V_i$.*

Then $s \leqslant n+1$. Further, the equality $s = n+1$ holds if and only if:

$$X = \mathbb{P}^n, \qquad V_i = \{X_i = 0\} \quad (1 \leqslant i \leqslant n+1)$$

(*in suitable projective coordinates*), *and f is given by*

$$f : [X_1, \ldots, X_{n+1}] \to \big[X_1^q, \ldots, X_{n+1}^q\big].$$

The conditions (1) and (2) in Theorem 1.1 are satisfied if X is smooth with Picard number $\rho(X) = 1$. The ampleness of V_i in Theorem 1.1 above and the related result Proposition 2.12 below (with the Cartier-ness of V_i replaced by the weaker $\mathbb{Q}$-Cartier-ness) is quite necessary because there are endomorphisms f of degree > 1 on toric surfaces whose boundary divisors have as many irreducible components as you like and are all stabilized by f^{-1}. The condition (1) is used to guarantee the inversion of adjunction (cf. [14]) and can be removed in dimension two (cf. [9, Theorem B], [25, Theorem 4.3.1]).

A projective variety X is *rationally chain connected* if every two points $x_i \in X$ are contained in a connected chain of rational curves on X. When X is smooth, X is rationally *chain* connected if and only if X is *rationally connected*, in the sense of Campana, and Kollár–Miyaoka–Mori [4,18].

The condition (1) below is satisfied if X is $\mathbb{Q}$-factorial with Picard number $\rho(X) = 1$, while the smoothness (or at least the mildness of singularities) of X in (3) is necessary (cf. Remark 1.8).

Theorem 1.2. [2] *Let X be a normal projective variety of dimension $n \geqslant 2$, $f : X \to X$ an endomorphism of degree > 1, $(0 \neq)$ $V = \sum_i V_i \subset X$ a reduced divisor with $f^{-1}(V) = V$, and $H \subset X$ an ample Cartier divisor. Assume the three conditions below* (*for all* i):

(1) *$-K_X \sim_{\mathbb{Q}} rH$ ($\mathbb{Q}$-linear equivalence) and $V_i \sim_{\mathbb{Q}} d_i H$ for some $r, d_i \in \mathbb{Q}$;*
(2) *X has only log canonical singularities around V; and*
(3) *X is further assumed to be smooth if: $V = V_1$ (i.e., V is irreducible), $K_X + V \sim_{\mathbb{Q}} 0$ and f is étale outside $V \cup f^{-1}(\operatorname{Sing} X)$.*

Then X, each irreducible component V_i and the normalization of V_i are all rationally chain connected. Further, $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor, i.e., $r > 0$ in (1).

A morphism $f : X \to X$ is *polarized* (by H) if

$$f^* H \sim qH$$

[2] By the recent paper of A. Broustet and A. Hoering "Singularities of varieties admitting an endomorphism," arXiv:1304.4013, the condition (1) in Theorem 1.1, condition (2) in Theorem 1.2 and similar conditions in Propositions 2.1 and 2.12 are automatically satisfied if X is $\mathbb{Q}$-Gorenstein and has a polarized endomorphism of degree > 1.