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Advances in Mathematics 252 (2014) 204–226

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Flat connections on configuration spaces and braid groups of surfaces

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Received 9 December 2011; accepted 25 October 2013

Available online 16 November 2013

Communicated by Roman Bezrukavnikov

Abstract

We construct an explicit bundle with flat connection on the configuration space of n points on a complex curve. This enables one to recover the '1-formality' isomorphism between the Lie algebra of the prounipotent completion of the pure braid group of n points on a surface and an explicitly presented Lie algebra, and to extend it to a morphism from the full braid group of the surface to the semidirect product of the associated group with the symmetric group S_n.

Keywords: Braid groups of surfaces; 1-Formality isomorphisms; Prounipotent completions; Flat connections; Knizhnik–Zamolodchikov–Bernard connections

0. Introduction

One of the achievements of rational homotopy theory has been a collection of results on fundamental groups of (quasi-)Kähler manifolds, leading in particular to insight on the Lie algebras of their prounipotent completions ([17,16,7]; for a survey see [1]). These results are particularly explicit in the case of configuration spaces $X = \mathrm{Cf}_n(M)$ of n distinct points on a manifold M [14,10,18]. In the particular case where M is a compact complex curve, they were made still more explicit in [5] (see also [13] for the case $M = \mathbb{C}$). In these works, a 'formality' isomorphism was established between this Lie algebra, denoted $\mathrm{Lie}\,\pi_1(X)$, and an explicit Lie algebra $\hat{\mathfrak{t}}_{g,n}$, where g is the genus of M ($\hat{\mathfrak{t}}_n$ when $M = \mathbb{C}$).

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0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.10.025

All these works take place in the framework of minimal model theory. However, alternative proofs are sometimes possible, based on explicit flat connections on X. Through the study of monodromy representations, such proofs allow for a deeper study of the algebra governing the formality isomorphisms, as well as for their connection to analysis and number theory.

In the case $X = \mathrm{Cf}_n(\mathbb{C})$, a construction of the formality isomorphism $\mathrm{Lie}\,\pi_1(X) \simeq \hat{\mathfrak{t}}_n$, based on a particular bundle with flat connection on X, can be extracted from [8]. This flat connection is at the basis of the theory of associators developed there; when certain Lie algebraic data are given, it specializes to the Knizhnik–Zamolodchikov connection [12]. When $X = \mathrm{Cf}_n(C)$, where C is an elliptic curve, a bundle with flat connection over X was constructed in [6] (see also [15]) and an isomorphism $\mathrm{Lie}\,\pi_1(X) \simeq \hat{\mathfrak{t}}_{1,n}$ was similarly derived; this flat connection specializes to the elliptic KZ–Bernard connection [3]. The corresponding analogue of the theory of associators was later developed by the author.

The goal of the present paper is to construct a similar explicit bundle with flat connection over $X = \mathrm{Cf}_n(C)$, C being a curve of genus $\geqslant 1$, and to derive from there an alternative construction of the isomorphism of [5]. We first recall this isomorphism (Section 1). We then recall some basic notions about bundles and flat connections in Section 2, and we formulate our main result: the construction of a bundle $\mathcal{P}_n$ over X with a flat connection α_{KZ} (Theorem 3), in Section 3. There we also show (Theorem 4) how this result enables one to recover the isomorphism result from [5], as well as to extend it to a morphism from the full braid group in genus g to $\exp(\hat{\mathfrak{t}}_{g,n}) \rtimes S_n$. Section 4 contains the explicit construction of the connection α_{KZ}. The sequel of the paper is devoted to the proof of its flatness. Section 5 is a preparation to this proof, and studies the behavior of α_{KZ} under certain simplicial homomorphisms. Section 6 contains the main part of the proof, while Section 7 contains the proof of some algebraic results on the Lie algebras $\mathfrak{t}_{g,n}$ which are used in the previous section. Finally, in Section 8, we relate the connection constructed in this paper to that constructed in [6] in genus one.

We hope to devote future work to applications of the present work to a theory of associators in genus g, as well as to relation with the higher genus KZB connection [4].

The author would like to thank D. Calaque and P. Etingof for collaboration in [6], as well as P. Humbert and G. Massuyeau for discussions.

1. Formality results

Let $g \geqslant 0$ and $n > 0$ be integers. The pure braid group with n strands in genus g is defined as $P_{g,n} := \pi_1(\mathrm{Cf}_n(S), x)$, where S is a compact topological surface of genus g without boundary, $\mathrm{Cf}_n(S) = S^n - (\text{diagonals})$ is the space of configurations of n points in S, and $x \in \mathrm{Cf}_n(S)$. The corresponding braid group is $B_{g,n} = \pi_1(\mathrm{Cf}_{[n]}(S), \{x\})$, where $\mathrm{Cf}_{[n]}(S) = \mathrm{Cf}_n(S)/S_n$ and $\{x\}$ is the S_n-orbit of x.

If $g > 0$ and $n \geqslant 0$, define $\mathfrak{t}_{g,n}$ as the $\mathbb{C}$-Lie algebra with generators[1] v^i ($v \in V$, $i \in [n]$), t_{ij} ($i \neq j \in [n]$), and relations[2]: $v \mapsto v^i$ is linear for $i \in [n]$,

$$[v^i, w^j] = \langle v, w\rangle t_{ij} \quad \text{for } i \neq j \in [n],\ v, w \in V,$$

$$\sum_{a=1}^{g} [x_a^i, y_a^i] = -\sum_{j: j\neq i} t_{ij}, \quad \forall i \in [n],$$

[1] We set $[n] := \{1, \dots, n\}$.

[2] The notation v^i does *not* stand for an ith power.