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Advances in Mathematics 252 (2014) 382–405

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

The higher Riemann–Hilbert correspondence

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Received 4 January 2013; accepted 4 November 2013

Available online 26 November 2013

Communicated by Tony Pantev

Abstract

We construct an A_∞-quasi-equivalence of dg-categories between $\mathcal{P}_{\mathcal{A}}$ — the category of prefect $\mathcal{A}^0$-modules with flat $\mathbb{Z}$-connection, associated to the de Rham dga $\mathcal{A}^\bullet$ of a compact manifold M — and the dg-category of *infinity-local systems* on M — homotopy-coherent representations of the smooth singular simplicial set of M. We understand this as a generalization of the classical Riemann–Hilbert correspondence to $\mathbb{Z}$-connections ($\mathbb{Z}$-graded superconnections in some circles). This theory makes crucial use of Igusa's notion of higher holonomy transport for $\mathbb{Z}$-connections which is a derivative of Chen's idea of generalized holonomy.

Keywords: Iterated integrals; Higher holonomy; Superconnections; dg-Categories; Riemann–Hilbert correspondence; Homotopy-coherent representations; Twisting cochains

1. Introduction

Given a closed, compact, connected manifold M, the classical Riemann–Hilbert correspondence (without singularities) gives an equivalence,

$$\operatorname{Reps}\bigl(\pi_1(M)\bigr) \cong \operatorname{Flat}(M),$$

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0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.11.001

between the category of representations of the fundamental group of M and the category of vector bundles with flat connection on M. The fact that this equivalence only depends on $\pi_1(M)$ is an indication that the classical theorem is a 1-truncation of a finer equivalence involving the entire homotopy-type of M. This paper develops such an untruncated Riemann–Hilbert correspondence.

We first associate to M two dg-categories, $\mathcal{P}_{\mathcal{A}}$ and $\mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_{\infty}M)$, which generalize the categories Flat(M) and Reps($\pi_1(M)$) respectively. The category $\mathcal{P}_{\mathcal{A}}$ is the dg-category of *modules with flat* $\mathbb{Z}$*-connection* or *cohesive modules* which was defined in [4]. We recount its definition at the start of Section 3. The objects can be regarded as $\mathbb{Z}$-graded vector bundles with homotopy-coherently-flat connections.

The second category, $\mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_{\infty}M)$ — defined in Section 2 — is the dg-category of *infinity-local systems*. The objects are maps of simplicial sets which to each smooth simplex in M assign a homotopy coherence in the category of chain complexes over $\mathbb{R}$.

Then we construct an A_∞-functor $\mathcal{RH}$ between these two dg-categories by virtue of a generalized notion of holonomy. The technology of iterated integrals suggests a precise and natural notion of such holonomy. Section 3 is devoted to defining and developing this concept. Ultimately the higher holonomy of a $\mathbb{Z}$-connection is a string of forms on the path space of M taking values in a particular endomorphism bundle. Such forms can be integrated over cycles in PM, and the flatness of the original connection implies that such a pairing induces an infinity-local system as desired. This process ultimately yields an A_∞-functor

$$\mathcal{RH}: \mathcal{P}_{\mathcal{A}} \to \mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_{\infty}M).$$

Our main result is the following:

Theorem 4.7. *The functor $\mathcal{RH}$ is an A_∞-quasi-equivalence.*

One recovers the classical Riemann–Hilbert theorem as a corollary after embedding the classical categories into each side and taking the 0th cohomology.

The main theorem is established by constructing $\mathcal{RH}$ and showing it is quasi-fully faithful and quasi-essentially surjective. The construction of $\mathcal{RH}$ relies on computing the curvature of an arbitrary $\mathbb{Z}$-connection and how such curvature is realized in the holonomy (Sections 3.3–3.5). Quasi-fully-faithfulness comes fairly easily in Section 4.2 by virtue of the de Rham theorem. Quasi-essential surjectivity requires considerably more work — the subject of Section 4.3.

In Section 5 we outline a couple non-trivial examples and extensions of our work.

2. Infinity-local systems

2.1. The definition of an infinity-local system

In this section we define a higher homotopical version of a local system which we call an *infinity-local system*. These objects are almost the same as the A_∞-functors of Igusa in [7], but are not required to have as a domain the nerve of a given category.

Now let K be a simplicial set and $\mathcal{C}$ an arbitrary dg-category over k a commutative ring. Fix a map $F: K_0 \to \mathcal{O}b\,\mathcal{C}$. Then define:

$$\mathcal{C}_F^{i,j} := \left\{\text{maps } f: K_i \to \mathcal{C}^j \;\middle|\; f(\sigma) \in \mathcal{C}^j\left(F(\sigma_{(i)}), F(\sigma_{(0)})\right)\right\},$$

and,