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Which traces are spectral?

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Abstract

Among ideals of compact operators on a Hilbert space we identify a subclass of those closed with respect to the logarithmic submajorization. Within this subclass, we answer the questions asked by Pietsch [22] and by Dykema, Figiel, Weiss and Wodzicki [7]. In the first case, we show that Lidskii-type formulae hold for every trace on such ideal. In the second case, we provide the description of the commutator subspace associated with a given ideal. Finally, we prove that a positive trace on an arbitrary ideal is spectral if and only if it is monotone with respect to the logarithmic submajorization.

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1. Introduction

Let H be a separable infinite-dimensional Hilbert space and let $\mathcal{L}(H)$ be the algebra of all bounded operators on H. The set $\mathcal{L}_1$ of all trace class operators is an ideal in $\mathcal{L}(H)$. It carries a special functional — the classical trace Tr. There is also the description of Tr as the sum of eigenvalues,

$$\mathrm{Tr}(T) = \sum_{n=0}^{\infty} \lambda(n, T), \qquad T \in \mathcal{L}_1. \tag{1}$$

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Here, $\lambda(T) = \{\lambda(n,T)\}_{n\geqslant 0}$ is the sequence of eigenvalues[1] of a compact operator T. This result was shown by von Neumann in [28] for self-adjoint operators and then by Lidskii in [20] in general case. Formula (1) is now known as Lidskii formula (see e.g. [25]).

Fix an orthonormal basis in the Hilbert space H. The subalgebra of $\mathcal{L}(H)$ consisting of all diagonal operators with respect to this basis is naturally isomorphic to the algebra l_∞ of all bounded complex sequences. Further, we always identify the algebra l_∞ with this diagonal subalgebra. Thus, the notations $x \in \mathcal{L}(H)$ (or $x \in \mathcal{I}$ for some ideal $\mathcal{I}$ in $\mathcal{L}(H)$) make perfect sense for an element $x \in l_\infty$.

Identifying the sequence $\lambda(T)$ with an element of $\mathcal{L}(H)$, we can write Lidskii formula as $\mathrm{Tr}(T) = \mathrm{Tr}(\lambda(T))$ for all $T \in \mathcal{L}_1$. A natural question concerning the extension of this formula to other ideals and traces on these ideals has been treated in a number of publications (see e.g. [2,3,10,11,15,17,21,22,24]). In what follows, $\mathcal{I}$ is a proper (algebraic) ideal in $\mathcal{L}(H)$ and φ is a trace on $\mathcal{I}$, i.e. a linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ satisfying the condition

$$\varphi(AB) = \varphi(BA), \quad A \in \mathcal{I},\ B \in \mathcal{L}(H).$$

Recall, that by the Calkin Theorem [4] the ideal $\mathcal{I}$ consists of compact operators only.

The following problem was stated by Pietsch (see p. 9 in [22]).

Question 1. *For which traces φ on an ideal $\mathcal{I}$ do we have*

$$\varphi(T) = \varphi\big(\lambda(T)\big), \quad T \in \mathcal{I}? \tag{2}$$

A given trace φ on the ideal $\mathcal{I}$ satisfying (2) is called spectral.

Study of traces in general and Question 1 in particular are closely related to the description of the commutator subspace of an ideal $\mathcal{I}$ in $\mathcal{L}(H)$. The latter subspace (denoted by $\mathrm{Com}(\mathcal{I})$) is a linear span of the elements $AB - BA$, $A \in \mathcal{I}$, $B \in \mathcal{L}(H)$. We emphasize that the latter concept is purely algebraic (without any norm or quasi-norm structure with which the ideal $\mathcal{I}$ may be endowed). The following question was asked in [7] (see also [8]).

Question 2. *Does the commutator subspace admit a description in spectral terms?*

Note that, for an operator $T \in \mathcal{I}$, we have $T \in \mathrm{Com}(\mathcal{I})$ if and only if all traces on $\mathcal{I}$ vanish on T. Thus, if Question 1 is answered in positive (in a sense that all traces on $\mathcal{I}$ are spectral) then, for $T \in \mathcal{I}$, we have $T \in \mathrm{Com}(\mathcal{I})$ if and only if $\lambda(T) \in \mathrm{Com}(\mathcal{I})$. Hence, a positive answer to Question 1 implies a positive answer to Question 2 and vice versa.

For normal operators, Question 2 was answered in the affirmative in [16] (see Theorem 3.1 there) and in [7] (see Theorem 5.6 there) for arbitrary ideals.

Theorem 3. *A normal operator $N \in \mathcal{I}$ belongs to* $\mathrm{Com}(\mathcal{I})$ *if and only if $C\lambda(N) \in \mathcal{I}$.*

Here, $C : l_\infty \to l_\infty$ is Cesaro operator defined by

$$Cx = \left(x(0), \frac{x(0)+x(1)}{2}, \frac{x(0)+x(1)+x(2)}{3}, \ldots\right), \quad x = \big(x(0), x(1), x(2), \ldots\big) \in l_\infty.$$

[1] Non-zero eigenvalues are repeated according to their algebraic multiplicity and arranged so that $\{|\lambda(n,T)|\}_{n\geqslant 0}$ is a decreasing sequence. If there are only finitely many (or none) non-zero eigenvalues, then all the other components of $\lambda(T)$ are zeros.