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Generalized dimensions of images of measures under Gaussian processes

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Abstract

We show that for certain Gaussian random processes and fields $X : \mathbb{R}^N \to \mathbb{R}^d$,

$$D_q(\mu_X) = \min\left\{d, \frac{1}{\alpha} D_q(\mu)\right\} \quad \text{a.s.},$$

for an index α which depends on Hölder properties and strong local nondeterminism of X, where $q > 1$, where D_q denotes generalized q-dimension and where μ_X is the image of the measure μ under X. In particular this holds for index-α fractional Brownian motion, for fractional Riesz–Bessel motions and for certain infinity scale fractional Brownian motions.

Keywords: Gaussian process; Local nondeterminism; Generalized dimension; Fractional Brownian

1. Introduction

Dimensions of images of sets under stochastic processes have been studied for many years. The Hausdorff dimension of the image or sample path of Brownian motion $X : \mathbb{R}_+ \to \mathbb{R}^d$ is almost surely equal to

$$\dim_H X(\mathbb{R}_+) = \min\{d, 2\},$$

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where $\dim_H$ denotes Hausdorff dimension, see Lévy [21], with the exact gauge function for the dimension established by Ciesielski and Taylor [6] for $d \geqslant 3$ and by Ray [27] and Taylor [32] for $d = 2$. Similar questions were subsequently studied for other processes, notably for sample paths of stable Lévy processes, see [33], and for fractional Brownian motion, see [1,2,17,23,30,31]. There are several comprehensive surveys of this work [2,17,34,37] which contain many further references.

A more general, but very natural, question is to find the almost sure dimensions of the image $X(E)$ of a Borel set $E \subseteq \mathbb{R}^N$ under a process $X : \mathbb{R}^N \to \mathbb{R}^d$, in terms of the dimension of E. In particular, Kahane [17] showed that

$$\dim_H X(E) = \min\left\{d, \frac{\dim_H E}{\alpha}\right\} \quad \text{a.s.} \tag{1.1}$$

if X is index-α fractional Brownian motion (which reduces to standard Brownian motion when $\alpha = \frac{1}{2}$ and $N = 1$).

The corresponding question for packing dimension $\dim_P$, where dimensions of images of sets can behave in a more subtle manner, was not answered until rather later, when Xiao [36] showed that for index-α fractional Brownian motion,

$$\dim_P X(E) = \frac{\dim_P^{\alpha d} E}{\alpha} \quad \text{a.s.,}$$

where $\dim_P^s E$ is the 'packing dimension profile' of E, a notion introduced in connection with linear projections of sets by Falconer and Howroyd [14], and which is defined in terms of a certain s-dimensional kernel.

In recent years, many other dimensional properties of the range, graph, level sets and images of given sets have been studied for a wide range of random processes, see [2,18,35,39,41] for surveys of this work.

It is natural to study dimensional properties of images of measures under random processes or fields in an analogous way to images of sets. For μ a Borel measure on $\mathbb{R}^N$ and $X : \mathbb{R}^N \to \mathbb{R}^d$, the random image measure μ_X on $\mathbb{R}^d$ is defined by

$$\mu_X(A) = \mu\{x\colon\ X(x) \in A\}, \quad A \subseteq \mathbb{R}^d.$$

When μ is the Lebesgue measure on $\mathbb{R}^N$ and X is a Gaussian process, the properties of the corresponding image measure μ_X have played important roles in studying the exact Hausdorff measure functions for the range, graph and level sets of X [30,35]. For more general Borel measures μ, one can look at the almost sure Hausdorff and packing dimensions of the measures (given by the minimal dimension of any set with complement of zero measure); indeed, by supporting suitable measures on sets, this approach is often implicit in the set dimension results mentioned above. Explicit results for Hausdorff and packing dimensions of image measures under a wide range of processes are given in [29], with dimension profiles again a key tool in the packing dimension cases.

However, the singularity structure of a measure may be very rich, and multifractal analysis in various forms has evolved to exhibit this structure as a function or spectrum; for general discussions see, for example, [12,16,20,25]. In this paper we consider generalized q-dimensions which reflect the asymptotic behavior as $r \searrow 0$ of the qth-moment sums $M_r(q) = \sum_C \mu(C)^q$ over the mesh cubes C of side r in $\mathbb{R}^N$. It will be convenient for us to work with the equivalent qth-moment integrals $\int \mu(B(x,r))^{q-1}\, d\mu(x)$, where $B(x,r)$ is the ball with center x and radius r, see Section 2 or [20,24] for further details of q-dimensions.