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Sabitov polynomials for volumes of polyhedra in four dimensions

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Abstract

In 1996 I.Kh. Sabitov proved that the volume of a simplicial polyhedron in a 3-dimensional Euclidean space is a root of certain monic polynomial with coefficients depending on the combinatorial type and on edge lengths of the polyhedron only. Moreover, the coefficients of this polynomial are polynomials in edge lengths of the polyhedron. This result implies that the volume of a simplicial polyhedron with fixed combinatorial type and edge lengths can take only finitely many values. In particular, this yields that the volume of a flexible polyhedron in a 3-dimensional Euclidean space is constant. Until now it has been unknown whether these results can be obtained in dimensions greater than 3. In this paper we prove that all these results hold for polyhedra in a 4-dimensional Euclidean space.

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1. Introduction

In 1996 I.Kh. Sabitov [17] proved that the volume of a (not necessarily convex) simplicial polyhedron in $\mathbb{R}^3$ is a root of certain monic polynomial whose coefficients are polynomials in the squares of edge lengths of the polyhedron. To give rigorous formulation of this result we need to specify what is understood under a simplicial polyhedron. Let K be a triangulation of a closed oriented surface. Then an *oriented polyhedron* (or a *polyhedral surface*) of combinatorial type K is a mapping $P : K \to \mathbb{R}^3$ whose restriction to every simplex of K is linear. Notice that we do not require P to be an embedding, so we allow a polyhedron to be degenerate and self-intersected.

By $[uvw]$ we denote the oriented triangle of K with vertices u, v, and w, and the orientation given by the prescribed order of vertices. Then the triangles $[uvw]$, $[vwu]$, and $[wuv]$ coincide and $[vuw]$ is the same triangle with the opposite orientation. An oriented triangle $[uvw]$ is said to be positively oriented if its orientation coincides with the given orientation of K. The set of all positively oriented triangles of K will be denoted by K_+. For points $A_0, A_1, A_2, A_3 \in \mathbb{R}^3$, we denote by $[A_0A_1A_2A_3]$ the oriented tetrahedron with vertices A_0, A_1, A_2, A_3 and by $V([A_0A_1A_2A_3])$ its oriented volume. (Notice that the tetrahedron $[A_0A_1A_2A_3]$ may be degenerate.)

Choose an arbitrary base point $O \in \mathbb{R}^3$. The *generalized volume* of a polyhedron $P : K \to \mathbb{R}^3$ is defined by

$$V(P) = \sum_{[uvw]\in K_+} V\big(\big[O\,P(u)P(v)P(w)\big]\big).$$

Obviously, the generalized volume of P is independent of the choice of the base point $O \in \mathbb{R}^3$.

Remark 1.1. If $P : K \to \mathbb{R}^3$ is an embedding, then $V(P)$ coincides up to sign with the usual volume of the domain in $\mathbb{R}^3$ bounded by the surface $P(K)$. So in a general case the generalized volume $V(P)$ should be understood as the volume of some "generalized domain" bounded by the singular surface $P(K)$. This can be made rigorous in the following way. We have

$$V(P) = \int_{\mathbb{R}^3} \lambda(x)\,dx, \tag{1}$$

where $dx = dx_1\,dx_2\,dx_3$ is the standard measure in $\mathbb{R}^3$ and $\lambda(x)$ is the algebraic intersection number of an arbitrary curve in $\mathbb{R}^3$ going from x to the infinity, and the oriented singular surface $P(K)$. (Obviously, λ is a piecewise constant function defined off the surface $P(K)$.) Thus a polyhedron $P : K \to \mathbb{R}^3$ should be thought about as a three-dimensional object "bounded by the singular surface $P(K)$" rather than as a two-dimensional object.

If $[uv]$ is an edge of K, then we denote by $l_{uv}(P)$ the square of the length of the edge $[P(u)P(v)]$ of a polyhedron $P : K \to \mathbb{R}^3$. By $l(P)$ we denote the set of all squared edge lengths $l_{uv}(P)$, where $[uv] \in K$.

Recall that a polynomial is called *monic* if its leading coefficient is equal to 1.

Theorem 1.1. *(See I.Kh. Sabitov [17].) Let K be a triangulation of a closed oriented surface. Then there exists a monic polynomial*