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Fibrations of topological stacks

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Abstract

In this note we define fibrations of topological stacks and establish their main properties. When restricted to topological spaces, our notion of fibration coincides with the classical one. We prove various standard results about fibrations (long exact sequence for homotopy groups, Leray–Serre and Eilenberg–Moore spectral sequences, etc.). We prove various criteria for a morphism of topological stacks to be a fibration, and use these to produce examples of fibrations. We prove that every morphism of topological stacks factors through a fibration and construct the homotopy fiber of a morphism of topological stacks. As an immediate consequence of the machinery we develop, we also prove van Kampen's theorem for fundamental groups of topological stacks.

Keywords: Fibration; Topological stack; Homotopy long exact sequence; Serre spectral sequence; Eilenberg–Moore spectra sequence; van Kampen's theorem

Contents

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1. Introduction

In this note we define fibrations of topological stacks and establish their main properties. Our theory generalizes the classical theory in the sense that when restricted to ordinary topological spaces our notion(s) of fibration reduce to the usual one(s) for topological spaces.

We prove some standard results for fibrations of topological stacks, e.g., the long exact sequence for homotopy groups (Theorem 5.2), the Leray–Serre spectral sequence (Theorems 7.6 and 7.7) and the Eilenberg–Moore spectral sequence (Theorem 8.1). We also prove that every morphism of topological stacks factors through a fibration (Theorem 6.1). We use this to define the homotopy fiber $\mathrm{hFib}(f)$ of a morphism $f:\mathcal{X}\to\mathcal{Y}$ of stacks (see Section 6).

van Kampen's theorem for the fundamental groups of topological stacks is also an immediate corollary of the results we prove in this paper (Theorem 5.10 and Corollary 5.11).

Since our fibrations are not assumed to be representable, it is often not easy to check whether a given morphism of stacks is a fibration straight from the definition. We prove the following useful local criterion for a map f to be a weak Serre fibration;[1] see Theorem 3.19.

Theorem 1.1. *Let $p:\mathcal{X}\to\mathcal{Y}$ be a morphism of topological stacks. If p is locally a weak Hurewicz fibration then it is a weak Serre fibration.*

In Section 4 we provide some general classes of examples of fibrations of stacks. Throughout the paper, we also prove various results which can be used to produce new fibrations out of old ones. This way we can produce plenty of examples of fibrations of topological stacks.

The results of this paper, though of more or less of standard nature, are not straightforward. The difficulty being that colimits are not well-behaved in the 2-category of topological stacks; for instance, gluing continuous maps along closed subsets does not always seem to be possible. For this reason, the usual methods for proving lifting properties (which involve building up

[1] For most practical purposes (e.g., constructing spectral sequences or the fiber homotopy exact sequence) weak fibrations are as good as fibrations.