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Uniformity of measures with Fourier frames

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Abstract

We examine Fourier frames and, more generally, frame measures for different probability measures. We prove that if a measure has an associated frame measure, then it must have a certain uniformity in the sense that the weight is distributed quite uniformly on its support. To be more precise, by considering certain absolute continuity properties of the measure and its translation, we recover the characterization on absolutely continuous measures $g\,dx$ with Fourier frames obtained in [24]. Moreover, we prove that the frame bounds are pushed away by the essential infimum and supremum of the function g. This also shows that absolutely continuous spectral measures supported on a set Ω, if they exist, must be the standard Lebesgue measure on Ω up to a multiplicative constant. We then investigate affine iterated function systems (IFSs), we show that if an IFS with no overlap admits a frame measure then the probability weights are all equal. Moreover, we also show that the Łaba–Wang conjecture [20] is true if the self-similar measure is absolutely continuous. Finally, we will present a new approach to the conjecture of Liu and Wang [29] about the structure of non-uniform Gabor orthonormal bases of the form $\mathcal{G}(g, \Lambda, \mathcal{J})$.

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Contents

1. Introduction

Everyone knows about Fourier series: the exponential functions $\{e^{2\pi inx}: n \in \mathbb{Z}\}$ form an orthonormal basis for $L^2[0, 1]$. Perturbations of the set $\mathbb{Z}$ will produce frames for $L^2[0, 1]$, or "non-harmonic" Fourier series, see e.g., [4,30]. This idea was later extended to orthonormal bases or frames of exponentials (Fourier frames) for fractal measures [6,13,18,16,17,21,28,33–35,7,8,12].

In [9] the notion of frames of exponentials for an arbitrary measure was extended to that of a frame measure.

Definition 1.1. Let μ be a finite, compactly supported Borel measure on $\mathbb{R}^d$. The *Fourier transform* of a function $f \in L^1(\mu)$ is defined by

$$\widehat{f\,d\mu}(t) = \int f(t)e^{-2\pi it\cdot x}\,d\mu(x) \quad \bigl(t \in \mathbb{R}^d\bigr).$$

Denote by e_t, $t \in \mathbb{R}^d$, the exponential function

$$e_t(x) = e^{2\pi it\cdot x} \quad \bigl(x \in \mathbb{R}^d\bigr).$$

We say that a Borel measure ν is a *Bessel measure* for μ if there exists a constant $B > 0$ such that for every $f \in L^2(\mu)$, we have

$$\|\widehat{f\,d\mu}\|^2_{L^2(\nu)} \leqslant B\|f\|^2_{L^2(\mu)}.$$

We call B a *(Bessel) bound* for ν. We say the measure ν is a *frame measure* for μ if there exist constants $A, B > 0$ such that for every $f \in L^2(\mu)$, we have

$$A\|f\|^2_{L^2(\mu)} \leqslant \|\widehat{f\,d\mu}\|^2_{L^2(\nu)} \leqslant B\|f\|^2_{L^2(\mu)}.$$

We call A, B *(frame) bounds* for ν. We call ν a *tight frame measure* if $A = B$ and *Plancherel measure* if $A = B = 1$.

Using the above definitions, we see that a set $E(\Lambda) := \{e_\lambda: \lambda \in \Lambda\}$ is a Fourier frame for $L^2(\mu)$ if and only if the measure $\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a frame measure for μ. $\{e_\lambda: \lambda \in \Lambda\}$ is a tight frame if and only if the measure $\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a tight frame measure for μ. When $E(\Lambda)$ is an orthonormal bases, μ is called a *spectral measure* and Λ is called a *spectrum* of μ [18,20].