Advances in Mathematics 258 (2014) 1–12

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Principle of local reflexivity respecting subspaces

Eve Oja [a,b,*]

[a] *Faculty of Mathematics and Computer Science, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia*
[b] *Estonian Academy of Sciences, Kohtu 6, 10130 Tallinn, Estonia*

ARTICLE INFO

Article history:
Received 26 June 2013
Accepted 4 March 2014
Available online 26 March 2014
Communicated by Dan Voiculescu

MSC:
primary 46B07
secondary 46B20, 46B28, 47B10

Keywords:
Principle of local reflexivity respecting subspaces
π_λ-(duality) property of a pair (Banach space its subspace)
Integral operators

ABSTRACT

We obtain a strengthening of the principle of local reflexivity in a general form. The added strength makes local reflexivity operators respect given subspaces. Applications are given to bounded approximation properties of pairs, consisting of a Banach space and its subspace.

1. Introduction and the main results

The principle of local reflexivity (PLR) is a powerful tool in the theory of Banach spaces and its applications. The PLR shows that the bidual X^{**} of a Banach space X is "locally" almost the same as the space X itself.

* Correspondence to: Faculty of Mathematics and Computer Science, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia.

E-mail address: eve.oja@ut.ee.

http://dx.doi.org/10.1016/j.aim.2014.03.007

Theorem 1.1 *(The PLR). (See [14] and [12] or, e.g., [11, p. 53].) Let X be a Banach space. If E and F are finite-dimensional subspaces of X^{**} and X^*, respectively, and $\varepsilon > 0$, then there exists a one-to-one linear operator $T : E \to X$ such that*

1° $\|T\|, \|T^{-1}\| < 1 + \varepsilon$,
2° $y^*(Tx^{**}) = x^{**}(y^*)$ *for all* $x^{**} \in E$ *and* $y^* \in F$,
3° $Tx^{**} = x^{**}$ *for all* $x^{**} \in E \cap X$.

The PLR was discovered by Lindenstrauss and Rosenthal [14] in 1969. It was improved by Johnson, Rosenthal, and Zippin [12] in 1971. Since then, many new proofs, refinements, and generalizations of the PLR have been given in the literature (see, e.g., [3] and [19] for results and references).

Recently, the concept of the bounded approximation property of pairs was introduced and studied by Figiel, Johnson, and Pełczyński in the important paper [8]. This concept involves finite-rank operators fixing a subspace in a given Banach space (see Section 4). Studies in [8] and [20] seem to indicate a need for some kind of the PLR that would respect subspaces. In the present paper, we shall establish such versions of the PLR, see Theorems 1.2 and 1.3 below.

First, let us fix some (standard) notation. For Banach spaces X and Y, both real or both complex, we denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X to Y. And let $\mathcal{F}(X, Y)$ denote its subspace of finite-rank operators. If U is a subspace of X, then $U^\perp$ is its annihilator in the dual space X^*, i.e., $U^\perp = \{x^* \in X^*: x^*(x) = 0\ \forall x \in U\}$.

Theorem 1.2 *(PLR respecting subspaces). Let X and Y be Banach spaces, and let U and V be closed subspaces of X and Y, respectively. Let $S \in \mathcal{F}(X^{**}, Y^{**})$ satisfy $S(U^{\perp\perp}) \subset V^{\perp\perp}$. If E and F are finite-dimensional subspaces of X^{**} and Y^*, respectively, and $\varepsilon > 0$, then there exists an operator $T \in \mathcal{F}(X, Y)$ satisfying $T(U) \subset V$ such that*

1° $|\|T\| - \|S\|| < \varepsilon$,
2° $x^{**}(T^*y^*) = (Sx^{**})(y^*)$ *for all* $x^{**} \in E$ *and all* $y^* \in F$,
3° $T^{**}x^{**} = Sx^{**}$ *for all those* $x^{**} \in E$ *for which* $Sx^{**} \in Y$.

Moreover, if the restriction $S|_E$ is one-to-one, then also $T^{**}|_E$ is, and

1°° $\|(T^{**}|_E)^{-1}\| < \|(S|_E)^{-1}\| + \varepsilon$.

In the special case, when $X = Y$ and S is a projection, also T is a projection.

As an illustration how to apply Theorem 1.2, let us look at the probably most well-known variant of Bellenot's PLR [4]. This PLR asserts the same as Theorem 1.1 but, additionally, a closed subspace W of X is given and, correspondingly, the assertion