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Triangulations, orientals, and skew monoidal categories ☆

Stephen Lack *, Ross Street

Department of Mathematics, Macquarie University, NSW 2109, Australia

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ABSTRACT

A concrete model of the free skew-monoidal category Fsk on a single generating object is obtained. The situation is clubbable in the sense of G.M. Kelly, so this allows a description of the free skew-monoidal category on any category. As the objects of Fsk are meaningfully bracketed words in the skew unit I and the generating object X, it is necessary to examine bracketings and to find the appropriate kinds of morphisms between them. This leads us to relationships between triangulations of polygons, the Tamari lattice, left and right bracketing functions, and the orientals. A consequence of our description of Fsk is a coherence theorem asserting the existence of a strictly structure-preserving faithful functor $\mathrm{Fsk} \longrightarrow \Delta_\perp$ where $\Delta_\perp$ is the skew-monoidal category of finite non-empty ordinals and first-element-and-order-preserving functions. This in turn provides a complete solution to the word problem for skew monoidal categories.

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* Corresponding author.
E-mail addresses: steve.lack@mq.edu.au (S. Lack), ross.street@mq.edu.au (R. Street).

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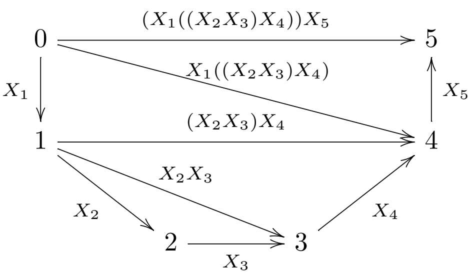
1. Introduction

Counting the number of triangulations of a convex polygon is a famous problem, a brief history of which can be found in [21, p. 212]. It seems that the problem is due to Euler, who proposed it to Segner. Segner gave a recurrence relation for the solution, and Euler gave the formula appearing on the left of the following equation

$$\frac{2}{2}\cdot\frac{6}{3}\cdot\frac{10}{4}\cdot\dots\cdot\frac{4n-2}{n+1}=\frac{1}{n+1}\frac{(2n)!}{n!n!}=\frac{1}{n+1}\binom{2n}{n}$$

for the number of triangulations of a convex polygon with $n+2$ vertices.

This can easily be transformed to the expressions on the right, whose values are now known as the Catalan numbers, and it seems to have been Catalan [4] who realized the equivalence between triangulations of a polygon and bracketings. The following diagram illustrates how a triangulation of a 6-gon provides a bracketing for a 5-fold product.



In the case of an associative multiplication, of course there is only one product of an ordered sequence of terms, but the combinatorics of such bracketings becomes significant in the context of non-associative multiplications.

Tamari [29] considered a partial order on the set of all such bracketings, where for bracketed words U, V, and W we have $(UV)W \leqslant U(VW)$, and where if $V \leqslant V'$ then $UV \leqslant UV'$ and $VW \leqslant V'W$. The poset Tam_n of all such bracketings of an n-fold product is in fact a lattice: this was proved in [6], but a more transparent proof was found in [7], using a combinatorial description of bracketings similar to the one we shall use below. See [20] for many articles related to Tamari's work, including its connections to the associahedra of Stasheff [22].

Mac Lane introduced the notion of monoidal category [17], which involves a functorial product, generally called the "tensor product", which need not be associative in the literal sense, but is associative up to natural isomorphism. Similarly there is a "unit object", which need not satisfy the usual unit laws in the literal sense, but does satisfy them up to natural isomorphism. These associativity and unit isomorphisms are required to satisfy five compatibility conditions, and Mac Lane showed that these five conditions imply, in a precise sense, that all diagrams built up using only these "structure isomorphisms" must commute. The fact that all diagrams commuted, in this sense, was summarized by