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Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions ☆

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ABSTRACT

We prove two monotonicity properties of $N(m, n)$, the number of partitions of n with rank m. They are (i) for any nonnegative integers m and n,

$$N(m,n) \geqslant N(m+2,n),$$

and, (ii) for any nonnegative integers m and n such that $n \geqslant 12$, $n \neq m+2$,

$$N(m,n) \geqslant N(m,n-1).$$

G.E. Andrews, B. Kim, and the first author introduced $\operatorname{ospt}(n)$, a function counting the difference between the first positive rank and crank moments. They proved that $\operatorname{ospt}(n) > 0$. In another article, K. Bringmann and K. Mahlburg gave an asymptotic estimate for $\operatorname{ospt}(n)$. The two monotonicity properties for $N(m, n)$ lead to stronger inequalities for $\operatorname{ospt}(n)$ that imply the asymptotic estimate.

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1. Introduction

A partition of an integer n is a nonincreasing sequence of positive integers whose sum is n, and the partition function, $p(n)$, counts the number of partitions of n. The rank of a partition, defined as the largest part minus the number of parts, was introduced by F.J. Dyson in 1944 [16]. Dyson conjectured that ranks of partitions gave combinatorial interpretations to Ramanujan's famous congruence for the partition function modulo 5 and 7, respectively. These conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [6] in 1954.

In [16], Dyson also conjectured the existence of another partition statistics, the cranks of partitions, that gives a combinatorial interpretation of Ramanujan's famous congruence for the partition modulo 11. The crank was later discovered by G.E. Andrews and F.G. Garvan [1].

Definition 1. For a partition π, let $\lambda(n)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0. \end{cases} \tag{1.1}$$

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leqslant 1$, we set [18] $M(0,0) = M(1,1) = M(-1,1) = 1$, $M(0,1) = -1$, $M(m,n) = 0$ otherwise. Let $M(r,m,n)$ denote the number the partitions of n with crank congruent to r modulo m. Similarly, let $N(m,n)$ count the number of partitions of n with rank m and $N(r,m,n)$ count the number of partitions of n with rank congruent to r modulo m.

Relations between ranks of partitions have been studied by several authors. Often, they are results on relations between $N(r,m,n)$ and $N(s,m,n)$, see for example, R. Lewis [26,29], Santa-Gadea [38], and K. Bringmann and B. Kane [8]. Many authors also studied relations between ranks and cranks of partitions. Many of these results give relations between $M(r,m,n)$ and $M(s,m,n)$, between $M(r,m,n)$ and $N(r,m,n)$, or a combination of them. See for example, [2,19,25,27,28,30,31]. Studies on the asymptotic behavior of ranks and cranks of partitions can be found in [7,8,13,22,23].

One objective of this paper is to establish monotonicity results on $N(m,n)$. It appears that these fundamental properties of $N(m,n)$ have been overlooked since their discovery in 1944. Our first result is the following inequality.

Theorem 2. *For any nonnegative integers m and n,*

$$N(m,n) \geqslant N(m+2,n).$$

We have the following strict inequalities.