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Cone-volume measures of polytopes

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Abstract

The cone-volume measure of a polytope with centroid at the origin is proved to satisfy the subspace concentration condition. As a consequence a conjectured (a dozen years ago) fundamental sharp affine isoperimetric inequality for the U-functional is completely established – along with its equality conditions.

MSC: 52A40; 52B11

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1. Introduction

Let $\mathcal{K}_o^n$ be the set of all convex bodies in $\mathbb{R}^n$ having the origin in their interiors, i.e., $K \in \mathcal{K}_o^n$ is a convex compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with $0 \in \operatorname{int}(K)$. For $K \in \mathcal{K}_o^n$ the *cone-volume measure*, V_K, of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel set $\omega \subseteq S^{n-1}$ by

$$\mathrm{V}_K(\omega) = \frac{1}{n} \int\limits_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle \, \mathrm{d}\mathcal{H}^{n-1}(x), \tag{1.1}$$

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where $\nu_K : \mathrm{bd}' K \to S^{n-1}$ is the Gauss map of K, defined on $\mathrm{bd}' K$, the set of points of the boundary of K having a unique outer normal, $\langle x, \nu_K(x)\rangle$ is the standard inner product on $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., [2,4,5,9,16,17,20–22,28].

In particular, in the very recent and groundbreaking paper [5] on the logarithmic Minkowski problem, Böröczky Jr., Lutwak, Yang and Zhang characterize the cone-volume measures of origin-symmetric convex bodies as exactly those non-zero finite even Borel measures on S^{n-1} which satisfy the *subspace concentration condition*. Here a finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration condition* if for every subspace $L \subseteq \mathbb{R}^n$

$$\mu\big(L \cap S^{n-1}\big) \leqslant \frac{\dim L}{n}\mu\big(S^{n-1}\big), \tag{1.2}$$

and equality holds in (1.2) for a subspace L if and only if there exists a subspace $\overline{L}$, complementary to L, so that also

$$\mu\big(\overline{L} \cap S^{n-1}\big) = \frac{\dim \overline{L}}{n}\mu\big(S^{n-1}\big),$$

i.e., μ is concentrated on $S^{n-1} \cap (L \cup \overline{L})$.

This concentration condition is at the core of different problems in Convex Geometry; it provides not only the solution to the logarithmic Minkowski problem for origin-symmetric convex bodies [5], but, for instance, in [4, Theorem 1.2], it was shown that the subspace concentration condition is also equivalent to the property that a finite Borel measure has an affine isotropic image.

Now let $P \in \mathcal{K}^n_o$ be a polytope with facets $F_1, \dots, F_m$, and let $a_i \in S^{n-1}$ be the outer unit normal of the facet F_i, $1 \leqslant i \leqslant m$. For each facet we consider $C_i = \mathrm{conv}\{0, F_i\}$, i.e., the convex hull of F_i with the origin, or in other words, C_i is the cone/pyramid with basis F_i and apex 0.

The cone-volume measure of P is given by (cf. (1.1))

$$\mathrm{V}_P = \sum_{i=1}^{m} \mathrm{V}(C_i)\delta_{a_i},$$

where $\mathrm{V}(C_i)$ is the volume, i.e., n-dimensional Lebesgue measure, of C_i and δ_{a_i} denotes the delta measure concentrated on a_i. Hence, P satisfies the subspace concentration condition (cf. (1.2)) if for every subspace $L \subseteq \mathbb{R}^n$

$$\sum_{a_i \in L} \mathrm{V}(C_i) \leqslant \frac{\dim L}{n}\mathrm{V}(P), \tag{1.3}$$

and equality holds in (1.3) for a subspace L if and only if there exists a subspace $\overline{L}$, complementary to L, so that $\{a_j\colon a_j \notin L\} \subset \overline{L}$. In other words, $A = (A \cap L) \cup (A \cap \overline{L})$, where $A = \{a_1, \dots, a_m\}$.

In general, the cone-volume measure depends on the position of the origin and not every $K \in \mathcal{K}^n_o$ fulfills the subspace concentration condition. In order to extend results from the origin-symmetric case, in [4, Problem 8.9] it is asked whether the cone-volume measure of convex bodies having the centroid at the origin satisfies the subspace concentration condition and our main result gives an affirmative answer in the case of polytopes.