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The uniformization of certain algebraic hypergeometric functions

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Abstract

The hypergeometric functions ${}_nF_{n-1}$ are higher transcendental functions, but for certain parameter values they become algebraic, because the monodromy of the defining hypergeometric differential equation becomes finite. It is shown that many algebraic ${}_nF_{n-1}$'s, for which the finite monodromy is irreducible but imprimitive, can be represented as combinations of certain explicitly algebraic functions of a single variable; namely, the roots of trinomials. This generalizes a result of Birkeland, and is derived as a corollary of a family of binomial coefficient identities that is of independent interest. Any tuple of roots of a trinomial traces out a projective algebraic curve, and it is also determined when this so-called Schwarz curve is of genus zero and can be rationally parametrized. Any such parametrization yields a hypergeometric identity that explicitly uniformizes a family of algebraic ${}_nF_{n-1}$'s. Many examples of such uniformizations are worked out explicitly. Even when the governing Schwarz curve is of positive genus, it is shown how it is sometimes possible to construct explicit single-valued or multivalued parametrizations of individual algebraic ${}_nF_{n-1}$'s, by parametrizing a quotiented Schwarz curve. The parametrization requires computations in rings of symmetric polynomials.

MSC: 33C20; 33C80; 14H20; 14H45; 05E05

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1. Introduction

The hypergeometric functions ${}_nF_{n-1}(\zeta)$, $n \geqslant 1$, are parametrized special functions of fundamental importance. Each ${}_nF_{n-1}(\zeta)$ is a function of a single complex variable, and in general is a higher transcendental function. It is parametrized by complex numbers $a_1, \dots, a_n; b_1, \dots, b_{n-1}$, and is written as ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$. It is analytic on $|\zeta| < 1$, with definition

$$ {}_nF_{n-1}\left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix} \;\middle|\; \zeta\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_{n-1})_k (1)_k} \zeta^k. \tag{1.1} $$

Here $(c)_k := c(c+1)\cdots(c+k-1)$, and the lower parameters $b_1, \dots, b_{n-1}$ may not be non-positive integers. The $n = 1$ function ${}_1F_0(a_1; -; \zeta)$ equals $(1-\zeta)^{-a_1}$, and the $n = 2$ function ${}_2F_1(a_1, a_2; b_1; \zeta)$ is the Gauss hypergeometric function.

If its parameters are suitably chosen, ${}_nF_{n-1}(\zeta)$ will become an *algebraic* function of ζ. Equivalently, if one regards ${}_nF_{n-1}$ as a single-valued function on a certain Riemann surface, defined by continuation from the disk $|\zeta| < 1$, then the surface will become compact. If $n = 1$, this occurs when $a_1 \in \mathbb{Q}$. In the first nontrivial case $n = 2$, the characterization of the triples $(a_1, a_2; b_1)$ for which ${}_2F_1(a_1, a_2; b_1; \zeta)$ is algebraic is a classical result of Schwarz. There is a finite list of possible *normalized* triples (the famous 'Schwarz list'), and ${}_2F_1$ is algebraic iff $(a_1, a_2; b_1)$ is a denormalized version of a triple on the list. Denormalization involves integer displacements of the parameters. For specifics, see [10, §2.7.2].

More recently, Beukers and Heckman [3] treated $n \geqslant 3$, and obtained a complete characterization of the parameters $(a_1, \dots, a_n; b_1, \dots, b_{n-1})$ that yield algebraicity. Like the $n = 2$ result of Schwarz, their result was based on the fact that the function ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$ satisfies an order-n differential equation on the Riemann sphere $\mathbb{P}^1_\zeta$, called $E_n(a_1, \dots, a_n; b_1, \dots, b_{n-1}, 1)_\zeta$ below. In modern language, E_n specifies a flat connection on an n-dimensional vector bundle over $\mathbb{P}^1_\zeta$. It has singular points at $\zeta = 0, 1, \infty$, and its (projective) monodromy group is generated by loops about these three points. The monodromy, resp. projective monodromy group is a subgroup of $GL_n(\mathbb{C})$, resp. $PGL_n(\mathbb{C})$, and algebraicity occurs iff the monodromy is finite. Schwarz exploited the classification of the finite subgroups of $PGL_2(\mathbb{C})$. In a *tour de force*, Beukers and Heckman handled the $n \geqslant 3$ case by exploiting the Shephard–Todd classification of the finite subgroups of $GL_n(\mathbb{C})$ generated by complex reflections. Their characterization result, however, is non-constructive: it supplies an algorithm for determining whether a given function ${}_nF_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; \zeta)$ is algebraic in ζ, but it does not yield a polynomial equation (with coefficients polynomial in ζ) satisfied by the function.

In this paper, a new approach is taken to the problem of constructing equations satisfied by algebraic ${}_nF_{n-1}$'s. Several classes of hypergeometric function, known to be algebraic by Beukers–Heckman, are made explicit by being *uniformized*. Recall that an algebraic function F may be of genus zero, i.e., may have a 'uniformization,' or parametrization, by rational functions. That is, one may have $F(R_1(t)) = R_2(t)$ for certain rational functions R_1, R_2, so that formally, $F = R_2 \circ R_1^{-1}$. In this case the Riemann surface of $F = F(\zeta)$, on which ζ and F are single-valued meromorphic functions, is isomorphic to the Riemann sphere $\mathbb{P}^1_t$. A sample result of this nature, obtained below, is the following. Let $n = p + 2$, where $p \geqslant 1$ is odd. Then

$$ {}_nF_{n-1}\left(\begin{matrix} \frac{a}{n}, \dots, \frac{a+(n-1)}{n} \\ \frac{a+1}{p}, \dots, \frac{a+p}{p}, \frac{1}{2} \end{matrix} \;\middle|\; \frac{4n^n}{p^p} \frac{t^2(1-t^2)^{2p}[(1+t)^p + (1-t)^p]^p}{[(1+t)^n + (1-t)^n]^n}\right) $$