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Properly immersed submanifolds in complete Riemannian manifolds

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Abstract

We consider a properly immersed submanifold M in a complete Riemannian manifold N. Assume that the sectional curvature K^N of N satisfies $K^N \geqslant -L(1+\mathrm{dist}_N(\cdot, q_0)^2)^{\frac{\alpha}{2}}$ for some $L > 0$, $2 > \alpha \geqslant 0$ and $q_0 \in N$. If there exists a positive constant $k > 0$ such that $\Delta|\mathbf{H}|^2 \geqslant k|\mathbf{H}|^4$, then we prove that M is minimal. We also obtain similar results for totally geodesic submanifolds. Furthermore, we consider a properly immersed submanifold M in a complete Riemannian manifold N with $K^N \geqslant -L(1+\mathrm{dist}_N(\cdot, q_0)^2)^{\frac{\alpha}{2}}$ for some $L > 0$, $2 > \alpha \geqslant 0$ and $q_0 \in N$. Let u be a smooth non-negative function on M. If there exists a positive constant $k > 0$ such that $\Delta u \geqslant ku^2$, and $|\mathbf{H}| \leqslant C(1+\mathrm{dist}_N(\cdot, q_0)^2)^{\frac{\beta}{2}}$ for some $C > 0$ and $1 > \beta \geqslant 0$, then we prove that $u = 0$ on M. By using the above result, we show that a non-negative biminimal properly immersed submanifold M in a complete Riemannian manifold N with $0 \geqslant K^N \geqslant -L(1+\mathrm{dist}_N(\cdot, q_0)^2)^{\frac{\alpha}{2}}$ is minimal. These results give affirmative partial answers to the global version of generalized Chen's conjecture for biharmonic submanifolds.

Keywords: Properly immersed submanifolds; Biharmonic maps; Biharmonic submanifolds; Generalized Chen's conjecture; Minimal submanifolds

1. Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. Harmonic maps between two Riemannian manifolds are critical points of the energy functional $E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \, dv_g$, for smooth maps $\phi : (M^m, g) \to (N^n, h)$ from an m-dimensional

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Riemannian manifold into an n-dimensional Riemannian manifold, where dv_g denotes the volume element of g. The Euler–Lagrange equation of E is

$$\tau(\phi) := \operatorname{Trace} \nabla d\phi = 0,$$

where $\tau(\phi)$ is called the *tension field* of ϕ.

On the other hand, in 1983, J. Eells and L. Lemaire [15] proposed the study of *polyharmonic maps of order* k. In 1986, G.Y. Jiang [17] studied *biharmonic maps* (that is, polyharmonic maps of order 2) which are critical points of the *bi-energy functional*

$$E_2(\phi) := \frac{1}{2} \int_M \left|\tau(\phi)\right|^2 dv_g. \tag{1}$$

The Euler–Lagrange equation of E_2 is

$$\tau_2(\phi) := -\Delta^\phi \tau(\phi) - \sum_{i=1}^m R^N\bigl(\tau(\phi), d\phi(e_i)\bigr) d\phi(e_i) = 0,$$

where $\Delta^\phi := \sum_{i=1}^m (\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i})$, ∇^ϕ is the induced connection on the induced bundle $\phi^{-1}TN$, R^N is the Riemannian curvature of N i.e., $R^N(X,Y)Z := [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z$ for any vector fields X, Y and Z on N, and $\{e_i\}_{i=1}^m$ is a local orthonormal frame field on M. The section $\tau_2(\phi)$ in the induced bundle is called the *bi-tension field* of ϕ. Clearly, harmonic maps are biharmonic maps.

If an isometric immersion $\phi : (M, g) \to (N, h)$ is biharmonic, then M is called a *biharmonic submanifold* in N. In this case, we remark that the tension field $\tau(\phi)$ of ϕ is written as $\tau(\phi) = m\mathbf{H}$, where $\mathbf{H}$ is the mean curvature vector field of M.

For biharmonic submanifolds, there is an interesting problem, namely Chen's conjecture (cf. [8]).

Conjecture 1. *Any biharmonic submanifold in* $\mathbb{E}^n$ *is minimal.*

There are many affirmative partial answers to Conjecture 1 (cf. [8,9,13,14,16]). Conjecture 1 is solved completely if M is one of the following: (a) a curve [14], (b) a surface in $\mathbb{E}^3$ [8], (c) a hypersurface in $\mathbb{E}^4$ [13,16].

Note that, since there is no assumption of *completeness* for submanifolds in Conjecture 1, this is a problem of *local* differential geometry. Recently, Conjecture 1 was reformulated as a problem of *global* differential geometry (cf. [1,19,21,22]):

Conjecture 2. *Any* complete *biharmonic submanifold in* $\mathbb{E}^n$ *is minimal.*

On the other hand, Conjecture 1 was generalized as follows: *Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal* (cf. [2–6]). This generalization is also a problem of local differential geometry. Y.-L. Ou and L. Tang [23] gave a counterexample of this conjecture (see [17] for an affirmative answer). With these understandings, it is natural to consider the following conjecture.

Conjecture 3. *Any complete biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*