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A geometric inequality on hypersurface in hyperbolic space

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ABSTRACT

In this paper, we use the inverse curvature flow to prove a sharp geometric inequality on star-shaped and two-convex hypersurface in hyperbolic space.

1. Introduction

The classical Alexandrov–Fenchel inequalities for closed convex hypersurface $\Sigma \subset \mathbb{R}^n$ state that

$$\int_{\Sigma} \sigma_m(\kappa)\, d\mu \geqslant C_{n,m}\left(\int_{\Sigma} \sigma_{m-1}(\kappa)\, d\mu\right)^{\frac{n-m-1}{n-m}}, \quad 1 \leqslant m \leqslant n-1 \tag{1}$$

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where $\sigma_m(\kappa)$ is the m-th elementary symmetric polynomial of the principal curvatures $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ of Σ and $C_{n,m}$ is a universal constant. When $m = 0$, (1) is interpreted as the classical isoperimetric inequality

$$|\Sigma|^{\frac{1}{n-1}} \geqslant C_n \, Vol(\Omega)^{\frac{1}{n}}, \tag{2}$$

which holds on all bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\Sigma = \partial\Omega$. Here $|\Sigma|$ is the area Σ and C_n is a constant depending only on dimension n. Inequality (1) was generalized to star-shaped and m-convex hypersurface $\Sigma \subset \mathbb{R}^n$ by Guan and Li [8] using the inverse curvature flow recently, where m-convex means that the principal curvature of Σ lies in Garding's cone

$$\Gamma_m = \left\{ \kappa \in \mathbb{R}^{n-1} \;\middle|\; \sigma_i(\kappa) > 0,\ i = 1, \dots, m \right\}.$$

Recently, Huisken [11] showed that in the case $m = 1$, the assumption *star-shaped* can be replaced by *outward-minimizing*.

In this paper, we consider the hyperbolic space $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ endowed with the metric

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the unit sphere $\mathbb{S}^{n-1}$. It's a natural question to establish some analogue inequalities of (1) for closed hypersurface in $\mathbb{H}^n$. In the case of $m = 1$, $\sigma_1 = \sigma_1(\kappa)$ is just the mean curvature H of Σ. Gallego and Solanes [6] have obtained a generalization of (1) to convex hypersurface in hyperbolic space using integral geometric methods, however, their result does not seem to be sharp.

We say a closed hypersurface $\Sigma \subset \mathbb{H}^n$ is star-shaped if the unit outward normal ν satisfies $\langle \nu, \partial_r \rangle > 0$ everywhere on Σ, which is also equivalent to that Σ can be parametrized by a graph

$$\Sigma = \left\{ \big(r(\theta), \theta\big) \;\middle|\; \theta \in \mathbb{S}^{n-1} \right\}$$

for some smooth function r on $\mathbb{S}^{n-1}$. Denoting $\lambda(r) = \sinh r$, then $\lambda'(r) = \cosh r$. Recently, Brendle, Hung and Wang [3] proved the following sharp inequality for star-shaped and mean convex (i.e., $H > 0$) hypersurface $\Sigma \subset \mathbb{H}^n$:

$$\int\limits_{\Sigma} \left(\lambda' H - (n-1) \langle \bar{\nabla} \lambda', \nu \rangle \right) d\mu \geqslant (n-1) \omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}}, \tag{3}$$

where $|\Sigma|$ is the area of Σ and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Equality holds in (3) if and only if Σ is a geodesic sphere centered at the origin. We note that inequality (3) has some applications in general relativity, see [3,14]. In [4], de Lima and Girão also claimed a related inequality for hypersurfaces in hyperbolic space.