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Finitely strictly singular operators in harmonic analysis and function theory

Pascal Lefèvre [a,*,1], Luis Rodríguez-Piazza [b,1]

[a] *Univ Lille Nord de France, U-Artois, Laboratoire de Mathématiques de Lens EA 2462, Fédération CNRS Nord-Pas-de-Calais FR 2956, F-62 300 Lens, France*
[b] *Departamento de Análisis Matemático & IMUS, Facultad de Matemáticas, Universidad de Sevilla, Apartado de Correos 1160, 41 080 Sevilla, Spain*

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A B S T R A C T

We prove that the class of strictly singular operators and the class of finitely strictly singular operators are stable under perturbed domination. We prove a diagonal theorem, which is a vector valued version of the result due to V. Milman stating that the identity from ℓ^p to ℓ^q is finitely strictly singular when $p < q$. Some applications are given. The paper is illustrated with examples in harmonic analysis and function theory: we prove that the Fourier transform from $L^p(\mathbb{T})$ into $\ell^{p'}$ is finitely strictly singular if and only if $p \in (1,2)$. In the case $p = 1$, some examples of different kind of behavior involving thin sets are given. Another result deals with the identity from the Hardy space H^p to the Bergman space $\mathcal{B}^{2p}$: it is finitely strictly singular for every $p \geqslant 1$. This result requires some new special inequalities between the norms on the Hardy and the Bergman spaces.

1. Introduction

Compactness of operators is a very powerful notion. Nevertheless, it is a strong property that occurs not so often. Several weaker forms were considered in functional analysis

* Corresponding author.
E-mail addresses: lefevre@euler.univ-artois.fr (P. Lefèvre), piazza@us.es (L. Rodríguez-Piazza).
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as weak compactness, complete continuity and strict singularity. They are all distinct in full generality. We are going to concentrate our attention on a quantified version of the notion of strict singularity: the finite strict singularity; and to pay attention to some special (classical) operators in the framework of harmonic analysis or theory of functions. Let us precise all these notions and their links.

It is well known that every compact operator is strictly singular and that the converse is false in general. More precisely

$$\text{compact} \quad \Longrightarrow \quad \text{finitely strictly singular} \quad \Longrightarrow \quad \text{strictly singular}$$

and each reverse implication is false in general.

The second implication is trivial (see the definitions below). We are going to precise the first implication. Let us recall that

Definition 1.1. An operator T from a Banach space X to a Banach space Y is Strictly Singular if it never induces an isomorphism on an infinite dimensional (closed) subspace of X: that is for every $\varepsilon > 0$ and every infinite dimensional subspace E of X, there exists x in the unit sphere of E such that $\|T(x)\| \leqslant \varepsilon$.

This notion is now very classical and widely studied: see [12, p. 75] or [11] for instance to know more on this notion.

Definition 1.2. An operator T from a Banach space X to a Banach space Y is Finitely Strictly Singular if: for every $\varepsilon > 0$, there exists $N_\varepsilon \geqslant 1$ such that for every subspace E of X with dimension greater than N_ε, there exists x in the unit sphere of E such that $\|T(x)\| \leqslant \varepsilon$.

This notion appears in the late sixties. For instance, in a paper of V. Milman [14], it is proved that the identity from ℓ^p to ℓ^q $(p < q)$ is finitely strictly singular. See [2,15,6] or [9] for recent results on this notion.

This can reformulated in terms of Bernstein approximation numbers (see [15] for instance). Recall that the n-th Bernstein number of an operator T is

$$b_n(T) = \sup_{\substack{E \subset X \\ \dim(E)=n}} \inf_{\substack{x \in E \\ \|x\|=1}} \|T(x)\|.$$

Hence, with this terminology, the operator T is Finitely Strictly Singular if and only if $(b_n(T))_{n \geqslant 1}$ belongs to the space c_0 of null sequences.

Let us precise why compact operators are finitely strictly singular (this is surely well known from the specialists).

Argument 1. Consider a norm one compact operator $T : X \to Y$. We suppose that T is not finitely strictly singular: there exists some $\varepsilon_0 > 0$ and for every $n \in \mathbb{N}$,