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On extendability by continuity of valuations on convex polytopes

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ABSTRACT

There is a well known construction of weakly continuous valuations on convex compact polytopes in $\mathbb{R}^n$. In this paper we investigate when a special case of this construction gives a valuation which extends by continuity in the Hausdorff metric to all convex compact subsets of $\mathbb{R}^n$. It is shown that there is a necessary condition on the initial data for such an extension. In the case of $\mathbb{R}^3$ more explicit results are obtained.

Contents

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0. Introduction

The aim of this paper is to discuss a well known construction of valuations on convex compact polytopes in $\mathbb{R}^n$ and study the question when the constructed valuations can be extended by continuity in the Hausdorff metric from polytopes to the class of all convex compact sets. We show that in general there is a non-trivial obstruction to such an extension. The case of 3-dimensional space is studied in greater detail, and we obtain necessary and sufficient conditions on the initial data of the construction under which the valuations do extend by continuity to all convex compact sets. The main results of the paper are Theorems 0.5, 0.7, 0.10 below.

Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{P}(V)$ denote the family of all convex compact polytopes in V, and let $\mathcal{K}(V)$ denote the family of all non-empty convex compact subsets in V.

Definition 0.1. A valuation on $\mathcal{P}(V)$ (resp. $\mathcal{K}(V)$) is a functional

$$\phi : \mathcal{P}(V) \longrightarrow \mathbb{C}$$
$$\big(\text{resp. } \phi : \mathcal{K}(V) \longrightarrow \mathbb{C}\big)$$

which is additive in the following sense: for any $A, B \in \mathcal{P}(V)$ (resp. $\mathcal{K}(V)$) such that the union $A \cup B$ is also convex, one has

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B).$$

As a generalization of the notion of measure, valuations on convex sets have long played an important role in geometry. Since the middle 1990s we have witnessed a breakthrough in the structure theory of valuations [28,36,1–4,6,7,13,14,30–33,39]. This progress in turn has led to immense advances in the integral geometry of isotropic (in particular, Hermitian) spaces (see [2,5,15,16,12]) and an understanding why certain classical notions from geometric analysis are indeed fundamental (see [23,32,30,31,38]).

Definition 0.2. A valuation ϕ on $\mathcal{K}(V)$ is called *continuous* if ϕ is continuous in the Hausdorff metric on $\mathcal{K}(V)$.

The notion of *weak continuity* of a valuation is often more appropriate for polytopes; it is due to Hadwiger [24]. Let us recall it. Fix $\xi := \{\xi_1, \dots, \xi_s\} \subset V^*$ an s-tuple of linear functionals on V. Let $\mathcal{P}_\xi$ denote the family of polytopes in $\mathcal{P}(V)$ of the form

$$P_\xi(y) = \big\{x \in V \bigm| \xi_i(x) \leqslant y_i \ \forall i = 1, \dots, s\big\}$$

where $y = (y_1, \dots, y_s) \in \mathbb{R}^s$ are such that $P_\xi(y)$ are non-empty and compact.