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Asymptotically liberating sequences of random unitary matrices

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ABSTRACT

A fundamental result of free probability theory due to Voiculescu and subsequently refined by many authors states that conjugation by independent Haar-distributed random unitary matrices delivers asymptotic freeness. In this paper we exhibit many other systems of random unitary matrices that, when used for conjugation, lead to freeness. We do so by first proving a general result asserting "asymptotic liberation" under quite mild conditions, and then we explain how to specialize these general results in a striking way by exploiting Hadamard matrices. In particular, we recover and generalize results of the second-named author and of Tulino, Caire, Shamai and Verdú.

1. Introduction

Of the results of Voiculescu [24,23] providing the foundations for free probability theory, arguably the simplest and most familiar is the following. Let $A^{(N)}$ and $B^{(N)}$ be

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deterministic N-by-N hermitian matrices with singular values bounded independently of N and having empirical distributions of eigenvalues tending in moments to limits μ_A and μ_B, respectively. Let $U^{(N)}$ be an N-by-N Haar-distributed random unitary matrix. Then the empirical distribution of eigenvalues of the sum $A^{(N)} + U^{(N)}B^{(N)}U^{(N)*}$ tends in moments to the free additive convolution $\mu_A \boxplus \mu_B$. The question addressed here, very roughly speaking, is this: how much less random can we make $U^{(N)}$ and still get free additive convolution in the limit? More generally, we ask: what sorts of random unitary matrices $U^{(N)}$ make A_N and $U^{(N)}B^{(N)}U^{(N)*}$ asymptotically free? Using the theory introduced here, we can show, for example, that the desired property of delivering asymptotic freeness is possessed by the random unitary matrix $U^{(N)} = W^{(N)*}\frac{H^{(N)}}{\sqrt{N}}W^{(N)}$ where $W^{(N)}$ is a uniformly distributed random N-by-N signed permutation matrix and $H^{(N)}$ is a deterministic N-by-N complex Hadamard matrix. (See Corollary 3.5 below.)

One precedent for our line of research is the main result of [13] which calculates the limiting distribution of singular values of a randomly chosen submatrix of the N-by-N discrete Fourier transform matrix $\mathrm{DFT}^{(N)}$. We can recover this result using our theory. (See Corollary 3.9 below.)

Another and much farther-reaching precedent is [21, Lemma 1, p. 1194]. This result is part of a study applying free probabilistic methods to problems of signal processing. A sample application of the result is the following. Let X and Y be bounded classical real random variables. Let $X^{(N)}$ and $Y^{(N)}$ be independent N-by-N diagonal matrices with diagonal entries that are i.i.d. copies of X and Y, respectively. Then $X^{(N)}$ and $\mathrm{DFT}^{(N)}Y^{(N)}\mathrm{DFT}^{(N)*}$ are asymptotically free. The latter result we can recover from our theory. (See Corollary 3.7 below.)

The notion of asymptotic freeness of Haar-distributed unitaries and other types of random or deterministic matrices has been extensively developed by many authors and in many directions. We just mention the papers [7,8] and [15] as particularly important influences on our work. The reader may consult, say, [1, Chap. 5], [16] or [24] for background and further references.

To the extent we make progress in this paper we do so by side-stepping issues of asymptotic freeness almost entirely. Instead we focus on the notion of *asymptotic liberation* (see Section 2.2) which is far easier to define and manipulate than asymptotic freeness. We mention in passing that the operator-theoretic paper [9] helped to push us toward a point of view emphasizing operators conjugation by which create freeness, and in particular we learned the term “liberation” from that source.

Another influence on the paper comes from applied mathematics, specifically the analysis of high-dimensional data. See for example the paper [20], which in the applied setting makes use of Hadamard matrices randomized both through random choice of block and randomization of signs through multiplication by diagonal matrices with i.i.d. diagonal entries of ± 1. We use a similar randomization to create arbitrarily large asymptotically liberating families from a single deterministic Hadamard matrix. (See Corollary 3.2 below.)