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Homotopy of area decreasing maps by mean curvature flow

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ABSTRACT

Let $f : M \to N$ be a smooth area decreasing map between two Riemannian manifolds (M, g_M) and (N, g_N). Under weak and natural assumptions on the curvatures of (M, g_M) and (N, g_N), we prove that the mean curvature flow provides a smooth homotopy of f to a constant map.

1. Introduction

Given a continuous map $f : M \to N$ between two smooth manifolds M and N, it is an interesting problem to find canonical representatives in the homotopy class of f.

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One possible approach is the *harmonic map heat flow* that was defined by Eells and Sampson in [5]. If M and N carry appropriate Riemannian metrics, they proved long-time existence and convergence of the heat flow, showing that under these assumptions one finds harmonic representatives in a given homotopy class. This approach is applicable usually when the target space is negatively curved. However, in general one can neither expect long-time existence nor convergence of the flow, in particular for maps between spheres, since the flow usually develops singularities.

Another way to deform a smooth map $f: M \to N$ between Riemannian manifolds (M, g_M) and (N, g_N) is by deforming its corresponding graph

$$\Gamma(f) := \bigl\{\bigl(x, f(x)\bigr) \in M \times N\colon\ x \in M\bigr\},$$

in the product space $M \times N$ via the *mean curvature flow*. A graphical solution of the mean curvature flow can be described completely in terms of a smooth family of maps $f_t : M \to N$, $t \in [0, T)$, $f_0 = f$, where $0 < T \leqslant \infty$ is the maximal time for which the smooth graphical solution exists.

In the case of long-time existence of a graphical solution and convergence we would thus obtain a smooth homotopy from the map f to a *minimal map* $f_\infty : M \to N$ as time t tends to infinity. Recall that a map is called minimal, if and only if its graph is a minimal submanifold of $M \times N$.

The first result in this direction is due to Ecker and Huisken [4]. They proved long-time existence of entire graphical hypersurfaces in $\mathbb{R}^{n+1}$. Moreover, they proved convergence to a flat subspace, if the growth rate at infinity of the initial graph is linear. The crucial observation in their paper was that the scalar product of the unit normal with a height vector satisfies a nice evolution equation that can be used to bound the second fundamental form appropriately.

The complexity of the normal bundle in higher codimensions makes the situation much more complicated. Results analogous to that of Ecker and Huisken are not available any more without further assumptions. However, the ideas developed in the paper of Ecker and Huisken opened a new era for the study of the mean curvature flow of submanifolds in Riemannian manifolds of arbitrary dimension and codimension (see for example [19,17,18,3,12,15,13,16,9,8,2,1] and the references therein).

A map $f : M \to N$ is called *weakly length decreasing* if $f^*\mathrm{g}_N \leqslant \mathrm{g}_M$ and *strictly length decreasing*, if $f^*\mathrm{g}_N < \mathrm{g}_M$. Hence a length decreasing map has the property that its differential shortens the lengths of tangent vectors. A smooth map $f : M \to N$ is called *weakly area decreasing* if its differential $\mathrm{d}f$ decreases the area of two dimensional tangent planes, that is if

$$\bigl\|\mathrm{d}f(v) \wedge \mathrm{d}f(w)\bigr\|_{\mathrm{g}_N} \leqslant \|v \wedge w\|_{\mathrm{g}_M},$$

for all $v, w \in TM$. If the differential $\mathrm{d}f$ is strictly decreasing the area of two dimensional tangent planes, then f is called *strictly area decreasing*. Analogously, we may introduce the notion of *weakly* and *strictly k-volume decreasing maps*.