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Logarithmic bump conditions and the two-weight boundedness of Calderón–Zygmund operators ☆

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ABSTRACT

We prove that if a pair of weights (u, v) satisfies a sharp A_p-bump condition in the scale of all log bumps or certain loglog bumps, then Haar shifts map $L^p(v)$ into $L^p(u)$ with a constant quadratic in the complexity of the shift. This in turn implies the two weight boundedness for all Calderón–Zygmund operators. This gives a partial answer to a long-standing conjecture. We also give a partial result for a related conjecture for weak-type inequalities. To prove our main results we combine several different approaches to these problems; in particular we use many of the ideas developed to prove the A_2 conjecture. As a byproduct of our work we also disprove a conjecture by Muckenhoupt and Wheeden on weak-type inequalities for the Hilbert transform. This is closely related to the recent counterexamples of Reguera, Scurry and Thiele.

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1. Introduction

In this paper we prove several partial results related to a pair of long-standing conjectures in the theory of two-weight norm inequalities. To state the conjectures and our results we recall a few facts about Orlicz spaces; see [4, Chapter 5] for complete details. Given a Young function A, the complementary function $\bar{A}$ is the Young function that satisfies

$$t \leqslant A^{-1}(t)\bar{A}^{-1}(t) \leqslant 2t, \quad t > 0.$$

We will say that a Young function $\bar{A}$ satisfies the $B_{p'}$ condition, $1 < p < \infty$, if for some $c > 0$,

$$\int_c^\infty \frac{\bar{A}(t)}{t^{p'}} \frac{dt}{t} < \infty.$$

If A and $\bar{A}$ are doubling (i.e., if $A(2t) \leqslant CA(t)$, and similarly for $\bar{A}$), then $\bar{A} \in B_p$ if and only if

$$\int_c^\infty \left(\frac{t^p}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty.$$

Remark 1. As we will see with specific examples below, if $\bar{A} \in B_{p'}$, then $\bar{A}(t) \lesssim t^{p'}$ and $A(t) \gtrsim t^p$.

Given p, $1 < p < \infty$, let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. We say that the pair of weights (u, v) satisfies an A_p bump condition with respect to A and B if

$$\sup_Q \left\|u^{1/p}\right\|_{A,Q} \left\|v^{-1/p}\right\|_{B,Q} < \infty, \tag{1}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^d$, and the Luxemburg norm is defined by

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q A\big(|f(x)|/\lambda\big)\,dx \leqslant 1\right\}.$$

If (1) holds, then it is conjectured that

$$T : L^p(v) \to L^p(u). \tag{2}$$

Similarly, if the pair (u, v) satisfies the weaker condition