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Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space

Xiang Ma [a,*], Changping Wang [b], Peng Wang [c]

[a] *LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China*
[b] *College of Mathematics and Computer Sciences, Fujian Normal University, Fuzhou 350108, People's Republic of China*
[c] *Department of Mathematics, Tongji University, Shanghai 200092, People's Republic of China*

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Abstract

For spacelike stationary (i.e. zero mean curvature) surfaces in 4-dimensional Lorentz space, one can naturally introduce two Gauss maps and a Weierstrass-type representation. In this paper we investigate the global geometry of such surfaces systematically. The total Gaussian curvature is related with the surface topology as well as the indices of the so-called good singular ends by a Gauss–Bonnet type formula. On the other hand, as shown by a family of counterexamples to Osserman's theorem, finite total curvature no longer implies that Gauss maps extend to the ends. Interesting examples include the deformations of the classical catenoid, the helicoid, the Enneper surface, and Jorge–Meeks' k-noids. Each family of these generalizations includes embedded examples in the 4-dimensional Lorentz space, showing a sharp contrast with the 3-dimensional case.

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* Corresponding author. Fax: +86 10 62751801.
E-mail addresses: maxiang@math.pku.edu.cn (X. Ma), cpwang@fjnu.edu.cn (C. Wang), netwangpeng@tongji.edu.cn (P. Wang).

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1. Introduction

Zero mean curvature spacelike surfaces in 4-dimensional Lorentz space $\mathbb{R}^4_1$ include classical minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$ as special cases. They are no longer local minimizer or maximizer of the area functional. Hence we call them *stationary surfaces* which are always assumed to be spacelike in this paper.

We became interested in this topic when studying several surface classes arising from variational problems in Möbius geometry and Laguerre geometry [19,28]. After reducing our original problems to stationary surfaces in $\mathbb{R}^4_1$, we searched the literature and found very few papers on this general case, which contrasted sharply to the rich theory and deep results on minimal surfaces in $\mathbb{R}^3$ (see the recent survey [21]) and maximal surfaces in $\mathbb{R}^3_1$ [27].

This situation motivated us to extend the general theory about the global geometry and topology of minimal surfaces in $\mathbb{R}^3$ to these stationary surfaces in $\mathbb{R}^4_1$. As a preparation, in Section 2 we introduce the basic invariants, equations, and the Weierstrass-type representation formula in terms of two meromorphic functions ϕ, ψ (corresponding to two Gauss maps, namely the two lightlike normal directions) and a holomorphic 1-form $\mathrm{d}h$ (the height differential). These are quite similar to the theory of minimal surfaces in $\mathbb{R}^3$.

However, stationary surfaces have quite different global geometry. According to the classical Osserman's theorem, a complete minimal surface in $\mathbb{R}^3$ with finite total Gaussian curvature always has a well-defined limit of the Gauss map at each end. In contrast with this, in Section 3 we construct complete stationary surfaces with finite total curvature whose Gauss maps ϕ, ψ both have an essential singularity at one of the ends (hence the Weierstrass data could not extend analytically to the whole compactified Riemann surface).

Moreover, the behavior of the ends in $\mathbb{R}^4_1$ is more complicated. There might exist the so-called *singular end* where the limits of the two lightlike normal directions coincide. In terms of the Weierstrass data ϕ, ψ, at the end we have $\phi = \overline{\psi}$. When they take this limit value with the same