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Advances in Mathematics 248 (2013) 229–241

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Piecewise linear approximation of smooth functions of two variables ☆

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Received 20 June 2013; accepted 26 June 2013

Available online 30 August 2013

Communicated by Erwin Lutwak

Abstract

Given a piecewise linear (PL) function p defined on an open subset of $\mathbb{R}^n$, one may construct by elementary means a unique polyhedron with multiplicities $\mathbb{D}(p)$ in the cotangent bundle $\mathbb{R}^n \times \mathbb{R}^{n*}$ representing the graph of the differential of p. Restricting to dimension 2, we show that any smooth function $f(x, y)$ may be approximated by a sequence $p_1, p_2, \ldots$ of PL functions such that the areas of the $\mathbb{D}(p_i)$ are locally dominated by the area of the graph of df times a universal constant.

Keywords: PL approximation; Curvature integrals

1. Introduction

1.1. General introduction

In approximating smooth submanifolds by polyhedra, it is well known that pathologies may occur if the triangles of the polyhedra become too thin. For example, the famous "Schwartz lantern" (cf. [10], Section 3.1.3) shows that the areas of a sequence of polyhedra P_i approximating a surface S may fail to converge unless the ratio of the area of a general constituent triangle to the square of its diameter is bounded away from zero. On the other hand, if this "fatness" condition is imposed then the sequence enjoys convergence not only to first order (i.e. convergence

☆ Fu partially supported by NSF grant DMS-1007580. Scott supported by an NSF VIGRE Postdoctoral Fellowship.

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0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.06.031

of areas), but also to second order in the sense that certain curvature integrals (or, more precisely, their polyhedral analogues) for the P_i converge weakly to the corresponding integrals for S [2,6]. The key observation is that the corresponding *absolute* curvature integrals of the P_i remain bounded in the course of the approximation process.

However, there is a further subtlety: even if the fatness condition is imposed, the bound on the absolute curvature integrals of the P_i depends on the C^2 norm of S, which may be much larger than its absolute curvature integral (cf. Section 2.2 below).

In the present paper we show that by means of a more careful approach, the P_i may be chosen so that their absolute curvature integrals are bounded by universal constant multiples of the corresponding integrals for S. In order to isolate the analytic aspects of the process, instead of a smooth surface S we consider smooth and piecewise linear (PL) *functions* f defined on an open subset $U \subset \mathbb{R}^n$, replacing curvatures of S by invariants of the Hessian of f.

1.2. Statement of the main theorem

The curvature integrals of S that we consider are artifacts of the manifold $N(S) \subset \mathbb{R}^3 \times S^2$ of unit normals to S. Using the area formula, it is easy to see that the area of $N(S)$ is bounded above and below by constant multiples of $\int_S 1 + \|II\| + |\det II|$, where II is the second fundamental form. By the same token, the area of the graph of the differential df of a C^2 function is bounded above and below by constant multiples of $\int 1 + \|H_f\| + |\det H_f|$, where H_f is the Hessian (cf. (3)).

For certain classes of singular surfaces S and functions f, it is known that there exist integral currents $N(S)$ and $\mathbb{D}(f)$ that serve, respectively, as suitable replacements for the manifold of unit normals and the graph of the differential. We state the relevant theorem in the latter case.

Theorem 1.1. *(See [5,9].) Let $U \subset \mathbb{R}^n$ be an open set and let f be a function with locally integrable differential df (i.e. f belongs to the Sobolev space $W^{1,1}_{loc}(U)$). Then there is at most one integral current $\mathbb{D}(f)$ of dimension n in the cotangent bundle T^*U, with the properties*

- $\partial\mathbb{D}(f) = 0$,
- $\mathbb{D}(f)$ *is* **Lagrangian***, in the sense that $\mathbb{D}(f) \llcorner \omega = 0$, where ω is the canonical symplectic form of T^*U,*
- *if $K \Subset U$ then* **mass**$(\mathbb{D}(f) \llcorner \pi^{-1}K) < \infty$,
- *for any compactly supported continuous function $\phi : T^*U \to \mathbb{R}$*

$$\mathbb{D}(f)\big(\phi\pi^*(dx_1 \wedge \cdots \wedge dx_n)\big) = \int_{\mathbb{R}^n} \phi(x, df_x)\, d\mathcal{L}^n x. \tag{1}$$

Here $\pi : T^*U \to U$ is the projection. If $\mathbb{D}(f)$ exists then we say that f is a *Monge–Ampère (MA) function*, and refer to $\mathbb{D}(f)$ as the **gradient cycle** of f.

If $f \in C^{1,1}(U)$ (i.e. $f \in C^1$ and its derivative is locally Lipschitz) then $\mathbb{D}(f)$ is given by integration over the graph of the differential df, which is a Lipschitz submanifold of T^*U with orientation induced by the standard orientation of $U \subset \mathbb{R}^n$. Thus if f is not smooth then we may think of $\mathbb{D}(f)$ as a substitute for this graph, and its existence is an indication that f has good second order properties. It is not difficult (cf. [8]) to construct $\mathbb{D}(p)$ directly for a PL function p; for completeness, and to fix ideas, we give the construction in detail for $n = 2$ in Section 2.1