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# Quantum Pieri rules for tautological subbundles

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**Abstract**

We give quantum Pieri rules for quantum cohomology of Grassmannians of classical types, expressing the quantum product of Chern classes of the tautological subbundles with general cohomology classes. We derive them by showing the relevant genus zero, three-pointed Gromov–Witten invariants coincide with certain classical intersection numbers.


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## 1. Introduction

The complex Grassmannian $Gr(k, n+1)$ parameterizes $k$-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. It can be written as $X = G/P$ with $G$ being a complex Lie group of type $A$, i.e. $G = SL(n+1, \mathbb{C})$, and $P$ being a maximal parabolic subgroup of $G$. We will continue to call such $X$'s as *Grassmannians* even when $G$ is not of type $A$. Indeed when $G$ is a classical Lie group of type $B, C$ or $D$, such a Grassmannian parameterizes subspaces in a vector space which are isotropic with respect to a non-degenerate skew-symmetric or symmetric bilinear form.

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Therefore it is usually called an *isotropic Grassmannian*. Recall that the tautological subbundle $\mathcal{S}$ over any point $[V] \in Gr(k, n+1)$ is just the $k$-dimensional vector subspace $V$ itself. And it restricts to the tautological subbundle $\mathcal{S}$ of any isotropic Grassmannian.

The cohomology ring $H^*(X, \mathbb{Z})$ of an isotropic Grassmannian $X = G/P$, or more generally a generalized flag variety, has a natural basis consisting of Schubert cohomology classes $\sigma^u$, labeled by a subset of the Weyl group $W$ of $G$. The (small) quantum cohomology ring $QH^*(X)$ of $X$, as a vector space, is isomorphic to $H^*(X) \otimes \mathbb{Q}[t]$. The quantum ring structure is a deformation of the ring structure on $H^*(X)$ by incorporating three-pointed, genus zero Gromov–Witten invariants of $X$. Since $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, the homology class of a holomorphic curve in $X$ is labeled by its degree $d$. In the case of $X = IG(k, 2n)$ being a Grassmannian of type $C_n$, the Schubert cohomology classes $\sigma^u = \sigma^{\mathbf{a}}$ can also be labeled by *shapes* $\mathbf{a}$, which are certain pairs of partitions. Every nonzero Chern class $c_p(\mathcal{S}^*) = (-1)^p c_p(\mathcal{S}) = \sigma^p$ (up to a scale factor of 2) is then a special Schubert class given by a special shape $p$, and they generate the quantum cohomology ring $QH^*(IG(k, 2n))$. One of the main results of the present paper is the following formula.

**Quantum Pieri rule for tautological subbundles of $IG(k, 2n)$.** *(See Theorem 4.4.) For any shape* $\mathbf{a}$ *and every special shape $p$, in $QH^*(IG(k, 2n))$, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e(\mathbf{a},\mathbf{b})} \sigma^{\mathbf{b}} + t \sum 2^{e(\tilde{\mathbf{a}},\tilde{\mathbf{c}})} \sigma^{\mathbf{c}}.$$

Here $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{c}}$ are shapes associated to $\mathbf{a}$ and $\mathbf{c}$ respectively; $e(\mathbf{a}, \mathbf{b})$ and $e(\tilde{\mathbf{a}}, \tilde{\mathbf{c}})$ are cardinalities of certain combinatorial sets, determined by the classical Pieri rules of Pragacz and Ratajski [27]. We have also obtained similar formulas for Grassmannians of type $B$ and $D$, details of which are given in Section 4.

The aforementioned *quantum Pieri rule* is a quantum version of the classical Pieri rule for isotropic Grassmannians. The famous classical Pieri rules are known firstly for complex Grassmannians (see e.g. [15]). For $X = Gr(k, n+1)$, they describe the cup product of a general Schubert class in $H^*(X)$ with $c_p(\mathcal{S}^*)$ or $c_p(\mathcal{Q})$, where $\mathcal{Q}$ is the tautological quotient bundle over $X$ given by the exact sequence $0 \to \mathcal{S} \to \mathbb{C}^{n+1} \to \mathcal{Q} \to 0$. It was generalized for other partial flag varieties of type $A$, firstly given by Lascoux and Schützenberger [22], and was also generalized for Grassmannians $X$ of type $B, C$ or $D$. Note that there is also a tautological quotient bundle $\mathcal{Q}$ over $X$. When $X$ parameterizes maximal isotropic subspaces (roughly speaking) there is no difference between the Chern classes of $\mathcal{S}^*$ and $\mathcal{Q}$, and the classical Pieri rules has been given by Hiller and Boe [16]. When $X$ parameterizes non-maximal isotropic subspaces, the classical Pieri rules with respect to $c_p(\mathcal{S}^*)$ have been given by Pragacz and Ratajski [27,28], while the classical Pieri rules with respect to $c_p(\mathcal{Q})$ are just covered in the recent work of Buch, Kresch and Tamvakis [5] on quantum Pieri rules. In contrast to complex Grassmannians, knowing either of them cannot deduce the other one. There is also a previous work of Sertöz [29] as well as a generalized classical Pieri rule given by Bergeron and Sottile [1], which gives the formula for multiplying a Schubert class on a complete flag variety of type $B$ or $C$ by a special Schubert class pulled back from the Grassmannian of maximal isotropic subspaces.

The story of quantum Pieri rules are almost parallel to the story of the classical Pieri rules. The quantum Pieri rules are also known firstly for complex Grassmannians, which were firstly given by Bertram [2]. They were generalized by Ciocan-Fontanine [11] for other partial flag varieties of type $A$, and by Kresch and Tamvakis [19,20] for those $X$ that parameterize maximal isotropic subspaces. Recently in [5], Buch, Kresch and Tamvakis have given us the quantum Pieri rules with respect to $c_p(\mathcal{Q})$ for those $X$ that parameterize non-maximal isotropic subspaces. In contrast