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Central sets and substitutive dynamical systems

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Abstract

In this paper we establish a new connection between central sets and the *strong coincidence conjecture* for fixed points of irreducible primitive substitutions of Pisot type. Central sets, first introduced by Furstenberg using notions from topological dynamics, constitute a special class of subsets of ℕ possessing strong combinatorial properties: Each central set contains arbitrarily long arithmetic progressions, and solutions to all partition regular systems of homogeneous linear equations. We give an equivalent reformulation of the strong coincidence condition in terms of central sets and minimal idempotent ultrafilters in the Stone–Čech compactification $\beta\mathbb{N}$. This provides a new arithmetical approach to an outstanding conjecture in tiling theory, the *Pisot substitution conjecture*. The results in this paper rely on interactions between different areas of mathematics, some of which had not previously been directly linked: They include the general theory of combinatorics on words, abstract numeration systems, tilings, topological dynamics and the algebraic/topological properties of Stone–Čech compactification of ℕ.

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1. Introduction

An important open problem in the theory of substitutions is the so-called *strong coincidence conjecture*: It states that each pair of fixed points x and y of an irreducible primitive substitution of Pisot type are *strongly coincident*: There exist a letter a and a pair of Abelian equivalent words s, t, such that sa is a prefix of x and ta is a prefix of y. This combinatorial condition, originally due to P. Arnoux and S. Ito, is an extension of a similar condition considered by F.M. Dekking in [14] in the case of uniform substitutions. In this case Dekking proves that the condition is satisfied by the "pure base" of the substitution if and only if the associated substitutive subshift has *pure discrete spectrum*, i.e., is metrically isomorphic with translation on a compact Abelian group. The strong coincidence conjecture has been verified for irreducible primitive substitutions of Pisot type on a binary alphabet in [2] and is otherwise still open.

The strong coincidence conjecture is linked to diffraction properties of one-dimensional atomic arrangements in the following way. It is shown in [17] and [24] that an atomic arrangement determined by a substitution has pure point diffraction spectrum (i.e., is a perfect quasicrystal) if and only if the tiling system associated with the substitution has pure discrete dynamical spectrum. In a pair of seminal papers [26,27], G. Rauzy established a link between pure discreteness of the dynamical spectrum and the irreducible Pisot property for substitutions. The *Pisot substitution conjecture* asserts that the dynamical spectrum of the tiling system associated with an irreducible Pisot substitution has pure discrete dynamical spectrum. For the latter to hold, it is necessary, and conjecturally sufficient, for the substitution to satisfy the strong coincidence condition.

In this paper we establish a link between the strong coincidence conjecture and central sets, originally introduced by Furstenberg in [19]. More precisely, we obtain an equivalent reformulation of the conjecture in terms of minimal idempotents in the Stone–Čech compactification $\beta\mathbb{N}$.

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ denote the set of natural numbers, and $\mathrm{Fin}(\mathbb{N})$ the set of all non-empty finite subsets of $\mathbb{N}$. A subset A of $\mathbb{N}$ is called an *IP-set* if A contains $\{\sum_{n\in F} x_n \mid F \in \mathrm{Fin}(\mathbb{N})\}$ for some infinite sequence of natural numbers $x_0 < x_1 < x_2 < \cdots$. A subset $A \subseteq \mathbb{N}$ is called an *IP*-set* if $A \cap B \neq \emptyset$ for every IP-set $B \subseteq \mathbb{N}$. In [19], Furstenberg introduced a special class of IP-sets, called central sets, having a substantial combinatorial structure. Central sets were originally defined in terms of topological dynamics:

Definition 1.1. A subset $A \subset \mathbb{N}$ is called *central* if there exists a compact metric space (X, d) and a continuous map $T : X \to X$, points $x, y \in X$ and a neighborhood U of y such that

- y is a uniformly recurrent point in X,
- x and y are proximal,
- $A = \{n \in \mathbb{N} \mid T^n(x) \in U\}$.

We say $A \subset \mathbb{N}$ is *central** if $A \cap B \neq \emptyset$ for every central set $B \subseteq \mathbb{N}$.

Recall that x is said to be *uniformly recurrent* in X if for every neighborhood V of x the set $\{n \mid T^n(x) \in V\}$ is syndetic, i.e., of bounded gap. Two points $x, y \in X$ are said to be *proximal* if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) < \epsilon$.

It is not evident from the above definition that central sets are IP-sets. The connection between the two lies in the algebraic and topological properties of the Stone–Čech compactification $\beta\mathbb{N}$.