Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 248 (2013) 495–530

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Spectral flow and the unbounded Kasparov product ☆

Jens Kaad [a,*], Matthias Lesch [b]

[a] *Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany*
[b] *Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany*

Received 11 October 2011; accepted 18 August 2013

Available online 7 September 2013

Communicated by Alain Connes

Abstract

We present a fairly general construction of unbounded representatives for the interior Kasparov product. As a main tool we develop a theory of C^1-connections on operator $*$-modules; we do not require any smoothness assumptions; our σ-unitality assumptions are minimal. Furthermore, we use work of Kucerovsky and our recent Local Global Principle for regular operators in Hilbert C^*-modules.

As an application we show that the Spectral Flow Theorem and more generally the index theory of Dirac–Schrödinger operators can be nicely explained in terms of the interior Kasparov product.

MSC: primary 19K35; secondary 46H25, 58J30, 46L80

Keywords: KK-theory; Operator modules; Kasparov product; Spectral flow

Contents

☆ Both authors were supported by the Hausdorff Center for Mathematics, Bonn.
* Corresponding author.
E-mail addresses: jenskaad@hotmail.com (J. Kaad), ml@matthiaslesch.de, lesch@math.uni-bonn.de (M. Lesch).
URLs: http://www.matthiaslesch.de, http://www.math.uni-bonn.de/people/lesch (M. Lesch).

0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.08.015

1. Introduction

The Spectral Flow Theorem ([28], or quite recently [15]) relates the spectral flow of a family, $A(x)$, of unbounded selfadjoint Fredholm operators to the index of the Fredholm operator $D = \frac{d}{dx} + A(x)$. D is an example of a so-called Dirac–Schrödinger operator on the complete manifold $\mathbb{R}$. Index theorems for such operators, at least in the special case where $A(x)$ is a finite rank bundle morphism, were established in the 80s and 90s, *e.g.*, [1,2] or [24, Chapter IV] and the references therein.

The family $\{A(x)\}_{x\in\mathbb{R}}$ naturally defines a class $[F_1]$ in the first K-theory group of $C_0(\mathbb{R})$, while the Dirac-operator $-i\frac{d}{dx}$ defines a class $[F_2]$ in the first K-homology group of the same C^*-algebra. It follows from KK-theory that the spectral flow of $A(x)$ can be recovered as the Kasparov product of the classes $[F_1]$ and $[F_2]$ via the natural identification $KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$, *cf.* [4, Section 18.10]. On the other hand classes in $KK(\mathbb{C},\mathbb{C})$ are represented by Fredholm operators and therefore the Spectral Flow Theorem can be rephrased in the following way: The Dirac–Schrödinger operator $D = \frac{d}{dx} + A(x)$, viewed as an unbounded $\mathbb{C}$–$\mathbb{C}$ Kasparov module represents the interior Kasparov product of $[F_1] \in K_1(C_0(\mathbb{R}))$ and $[F_2] \in K^1(C_0(\mathbb{R}))$.

It is tempting to generalize this pattern by replacing the real line by a complete Riemannian manifold. The family then becomes parametrized by the manifold whereas a Dirac operator (or, slightly more generally, a first order elliptic operator with bounded propagation speed) on the complete manifold naturally replaces $-i\frac{d}{dx}$. For the realization of this program it turns out that the existing theories of unbounded representatives for the KK-product, see *e.g.*, [3,21,26], do not suffice. It is the purpose of this paper to establish an appropriate improvement of unbounded KK-theory which naturally covers Dirac–Schrödinger operators on complete manifolds.

In the paper [26] Mesland develops a framework of smooth algebras and differentiable C^*-modules equipped with smooth connections with the purpose of establishing a general formula for the unbounded KK-product. We pursue a less technical approach: Since unbounded Kasparov modules are abstractions of *first order* elliptic differential operators it is most natural to impose C^1-conditions on both modules and connections. It turns out that the theory of operator