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Teichmüller theory and critically finite endomorphisms

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Abstract

We present a systematic way to generate critically finite endomorphisms of $\mathbb{P}^n$. These maps arise in the context of Teichmüller theory, specifically in Thurston's topological characterization of rational maps. The dynamical objects for the endomorphisms correspond to central objects from Thurston's theorem. Our theorems build infinitely many of these endomorphisms; in fact, a large number of examples of critically finite endomorphisms of $\mathbb{P}^n$ found in the literature arise from this construction.

Keywords: Complex dynamics; Critically finite endomorphisms of projective space; Teichmüller theory

0. Introduction

Let S^2 be an oriented topological 2-sphere. We begin with an orientation-preserving branched cover $f : (S^2, P) \to (S^2, P)$, where the domain and range spheres are identified, and P is the postcritical set of f. If $|P| < \infty$ then f is called a Thurston map. Each Thurston map induces a holomorphic endomorphism $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ where $\mathcal{T}_P$ is the Teichmüller space of the pair (S^2, P). Thurston's topological characterization of rational maps (Theorem 4.1) says that a Thurston map $f : (S^2, P) \to (S^2, P)$ is *combinatorially equivalent* to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ if and only if there are no *obstructing multicurves*. One proves Thurston's theorem by showing that the map f is equivalent to a rational map F if and only if the associated endomorphism $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ has a fixed point.

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This map σ_f rarely descends to the moduli space $\mathcal{M}_P$. However, sometimes an "inverse" of σ_f descends to yield a *moduli space map* $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, such that the following diagram commutes,

$$\begin{tikzcd} \mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi_P"'] & \mathcal{T}_P \arrow[d, "\pi_P"] \\ \mathcal{M}_P & \mathcal{M}_P \arrow[l, dashed, "g_f"'] \end{tikzcd}$$

where $\pi_P : \mathcal{T}_P \to \mathcal{M}_P$ is a holomorphic universal covering map. L. Bartholdi and V. Nekrashevych were the first to prove that moduli space maps exist by explicitly computing these maps for three examples in [1]. We generalize this result and prove that for certain classes of Thurston maps, a moduli space map exists. The moduli space map g_f is naturally defined on $\pi_P(\sigma_f(\mathcal{T}_P)) \subseteq \mathcal{M}_P$ which may only be a proper subset of moduli space.

If a moduli space map exists, we compactify $\mathcal{M}_P$ and ask if the map extends to the compactification. If f is a *topological polynomial*, then we use the $\mathbb{P}^n$ compactification of the moduli space. Theorem 5.17 asserts that if the Thurston map $f : (S^2, P) \to (S^2, P)$ is a topological polynomial with periodic critical points, then $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to a critically finite endomorphism $G_f : \mathbb{P}^n \to \mathbb{P}^n$. Theorem 5.18 asserts that if the Thurston map $f : (S^2, P) \to (S^2, P)$ is a unicritical topological polynomial, then $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ extends to a critically finite endomorphism $G_f : \mathbb{P}^n \to \mathbb{P}^n$.

In general, one obtains a correspondence on the level of moduli space. Determining whether the graph of this correspondence is the graph of a map g_f is rather difficult. We prove that it is actually a nondynamical issue as it depends on the *Hurwitz class* of f. We define the *Hurwitz space* $\mathcal{W}_f$ associated to f; we prove that it is a complex manifold and a finite cover of the moduli space $\mathcal{M}_P$ (Theorem 2.6).

Although the existence of a moduli space map is a nondynamical matter, we wish to exploit this map (when it exists) in a dynamical setting. As a general rule, the dynamical objects for $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ correspond to those for $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ and therefore also correspond to central objects in Thurston's theorem, Theorem 4.1. In this way, we can learn about $\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$ by examining the moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, and vice versa. In fact, applying the nondynamical discussion in the dynamical realm generalizes our setting: we sometimes apply our analysis to orientation-preserving branched covers $f : (S^2, P) \to (S^2, P)$, where P is a finite set that contains the postcritical set of f, and $f(P) \subseteq P$.

The critically finite endomorphisms $G_f : \mathbb{P}^n \to \mathbb{P}^n$ arising as moduli space maps have a variety of very nice properties. For example, the postcritical locus of each consists of $(n^2 + 3n + 2)/2$ hyperplanes (see Proposition 5.5). These hyperplanes comprise the boundary of $\mathcal{M}_P$ inside $\mathbb{P}^n$, so the complement of the postcritical locus of G_f in $\mathbb{P}^n$ is therefore Kobayashi hyperbolic (as a corollary of Royden's theorem [25]).

Critically finite endomorphisms of $\mathbb{P}^n$ were first studied by J.E. Fornæss and N. Sibony in [11]; part of their motivation was to provide a positive answer to a question asked by C. McMullen (see [2]). In addition to Fornæss and Sibony, several other mathematicians have studied critically finite endomorphisms of projective space. Among them are S. Crass [8], C. Dupont [10], M. Jonsson [16], T. Ueda [28]. Many of the existing examples of critically finite endomorphisms in the literature arise via our construction. For example, we can recover the original Fornæss and Sibony examples in [11] with our methods, in addition to the examples of Crass in [8]. There is one rather curious example that we cannot recover however; see Example 8.1 in Section 8.