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Multi-parameter singular Radon transforms II: The L^p theory

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Abstract

The purpose of this paper is to study the L^p boundedness of operators of the form

$$f \mapsto \psi(x) \int f\big(\gamma_t(x)\big) K(t)\, dt,$$

where $\gamma_t(x)$ is a C^∞ function defined on a neighborhood of the origin in $(t, x) \in \mathbb{R}^N \times \mathbb{R}^n$, satisfying $\gamma_0(x) \equiv x$, ψ is a C^∞ cut-off function supported on a small neighborhood of $0 \in \mathbb{R}^n$, and K is a "multi-parameter singular kernel" supported on a small neighborhood of $0 \in \mathbb{R}^N$. We also study associated maximal operators. The goal is, given an appropriate class of kernels K, to give conditions on γ such that every operator of the above form is bounded on L^p $(1 < p < \infty)$. The case when K is a Calderón–Zygmund kernel was studied by Christ, Nagel, Stein, and Wainger; we generalize their work to the case when K is (for instance) given by a "product kernel". Even when K is a Calderón–Zygmund kernel, our methods yield some new results. This is the second paper in a three part series. The first paper deals with the case $p = 2$, while the third paper deals with the special case when γ is real analytic.

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1. Introduction

The goal of this paper is to prove the L^p boundedness of (a special case of) the multi-parameter singular Radon transforms introduced in [17]. We consider operators of the form

$$T(f)(x) = \psi(x) \int f\big(\gamma_t(x)\big) K(t)\, dt, \tag{1.1}$$

where ψ is a C_0^∞ cut-off function (supported near, say, $0 \in \mathbb{R}^n$), $\gamma_t(x) = \gamma(t, x)$ is a C^∞ function defined on a neighborhood of the origin in $\mathbb{R}^N \times \mathbb{R}^n$ satisfying $\gamma_0(x) \equiv x$, and K is a "multi-parameter" distribution kernel, supported near 0 in $\mathbb{R}^N$. For instance, one could take K to be a "product kernel" supported near 0.[3] To define this notion, suppose we have decomposed $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu}$, and write $t = (t_1, \ldots, t_\nu) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu}$. A product kernel satisfies

$$\left|\partial_{t_1}^{\alpha_1} \cdots \partial_{t_\nu}^{\alpha_\nu} K(t)\right| \lesssim |t_1|^{-N_1-|\alpha_1|} \cdots |t_\nu|^{-N_\nu-|\alpha_\nu|},$$

along with certain "cancellation conditions".[4] The goal is to develop conditions on γ such that T is bounded on L^p $(1 < p < \infty)$. In addition, we will prove (under the same conditions on γ) L^p boundedness $(1 < p \leqslant \infty)$ for the corresponding maximal operator,

$$\mathcal{M} f(x) = \psi(x) \sup_{0<\delta_1,\ldots,\delta_\nu \leqslant a} \frac{1}{\delta_1^{N_1} \delta_2^{N_2} \cdots \delta_\nu^{N_\nu}} \int\limits_{|t_\mu| \leqslant \delta_\mu} \left|f\big(\gamma_t(x)\big)\right| dt_1 \cdots dt_\nu,$$

where $a > 0$ is some small number depending on γ. In fact, the L^p boundedness of $\mathcal{M}$ will be a step towards proving the L^p boundedness of T.

This paper is the second in a three part series. The first paper in the series [17] dealt with the L^2 theory, and applied to a larger class of kernels than the results in this paper do (see Section 2 for a discussion). However, all of the examples we have in mind (and in particular all of the examples discussed in [17]) do fall under the theory discussed in this paper. An L^2 result from [17] will serve as one of the main technical lemmas of this paper. The third paper in the series [15] deals with the special case when γ is assumed to be real analytic. In this case many of the assumptions from this paper take a much simpler form, and some of our results can be improved. See [14] for an overview of the series.

For a more detailed introduction to the operators defined in this paper, we refer the reader to [17], which discusses special motivating cases, and gives a number of examples.

Remark 1.1. The results in this paper generalize the L^p boundedness results from the work (in the single-parameter setting, when K is a Calderón–Zygmund kernel) of Christ, Nagel, Stein, and Wainger [2]. In fact, as discussed in [17], the results in this paper generalize the L^p boundedness results in [2] even if one considers only the single-parameter setting.[5] One major difficulty that

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[3] Our main theorem applies to classes of kernels other than product kernels.

[4] The simplest example of a product kernel is given by $K(t_1, \ldots, t_\nu) = K_1(t_1) \otimes \cdots \otimes K_\nu(t_\nu)$, where $K_1, \ldots, K_\nu$ are standard Calderón–Zygmund kernels. That is, K_j satisfies $|\partial_{t_j}^{\alpha_j} K_j(t_j)| \lesssim |t_j|^{-N_j-|\alpha_j|}$, again along with certain "cancellation conditions". When $\nu = 1$, the class of product kernels is precisely the class of Calderón–Zygmund kernels. See Section 16 of [17] for the statement of the cancellation conditions. We do not make them precise in this paper, since we will be working with more general kernels K.

[5] For instance, in the single-parameter setting if $K(t)$ is a Calderón–Zygmund kernel supported near $t = 0$, $\gamma_t(x)$ is a real analytic function in both variables, defined on a neighborhood of $(0, 0) \in \mathbb{R}^N \times \mathbb{R}^n$ satisfying $\gamma_0(x) \equiv x$, and