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The size of quadratic p-adic linearization disks

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Abstract

We find the exact radius of linearization disks at indifferent fixed points of quadratic maps in $\mathbb{C}_p$. We also show that the radius is invariant under power series perturbations. Localizing all periodic orbits of these quadratic-like maps we then show that periodic points are not the only obstruction for linearization. In so doing, we provide the first known examples in the dynamics of polynomials over $\mathbb{C}_p$ where the boundary of the linearization disk does not contain any periodic point.

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1. Introduction

Let p be a prime and let $\mathbb{C}_p$ be the completion of an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. We study a p-adic analogue of the Siegel center problem in complex dynamics. A power series

$$f(x) = \lambda x + \langle x^2 \rangle \in \mathbb{C}_p[[x]], \quad |\lambda| = 1, \text{ not a root of unity},$$

has an irrationally indifferent fixed point at $x = 0$, and is said to be analytically *linearizable* at $x = 0$ if there exists a convergent power series solution H_f to the functional equation

$$H_f \circ f \circ H_f^{-1}(x) = \lambda x.$$

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For a large class of quadratic-like maps we find the exact radius of the corresponding linearization disk about $x = 0$. Localizing the periodic orbits of these maps, we then show that the convergence of H_f stops *before* the appearance of any periodic point different from $x = 0$. Our starting point is Yoccoz study of quadratic Siegel disks [31]. Let α be irrational and $\{p_n/q_n\}_{n\geqslant 0}$ be the approximants given by its continued fraction expansion. The Brjuno series $B(\alpha)$ is defined by $B(\alpha) = \sum_{n\geqslant 0} \log(q_{n+1})/q_n$. Yoccoz proved that $P_\alpha(z) = e^{2i\pi\alpha}z + z^2 \in \mathbb{C}[z]$ is linearizable at $z = 0$ if and only if $B(\alpha) < \infty$. If $B(\alpha) < \infty$ and $C(\alpha)$ is the conformal radius of the Siegel disk, then there exist universal constants C_1 and C_2 such that $C_1 < B(\alpha) + \log C(\alpha) < C_2$. The lower bound is due to Yoccoz [31], and the upper bound is due to Buff and Chéritat [5]. A possible obstruction for linearization is the presence of periodic points different from zero. Indeed, Yoccoz proved that if α is irrational and $B(\alpha) = \infty$, then every neighborhood of $z = 0$ contains a periodic orbit of P_α different from $z = 0$. However, as shown by Pérez-Marco [23, Theorem V.4.2], in complex dynamics there exist maps having no periodic point on the boundary of a Siegel disk. Below, we present an analogue of the Yoccoz Brjuno function, $\tilde{r}(\lambda)$, that estimates the radius of p-adic linearization disks (Theorem A). Using the optimal bound obtained for p-adic quadratic maps (Theorem B), we obtain an analogue of Pérez-Marco's result on non-existence of periodic points on the boundary (Corollary C), providing the first p-adic examples of its kind.

Let p be a prime and denote by $\mathcal{O}_p$ the closed unit disk in $\mathbb{C}_p$. For any $\lambda \in \mathbb{C}_p$ with $|\lambda| = 1$ but λ not a root of unity, we define

$$\mathcal{G}_\lambda := \lambda x + x^2\mathcal{O}_p[[x]].$$

By the Non-Archimedean Siegel Theorem of Herman and Yoccoz [11], $f \in \mathcal{G}_\lambda$ is always linearizable at $x = 0$. Given $f \in \mathcal{G}_\lambda$, we denote by Δ_f the corresponding *linearization disk*, *i.e.* the maximal disk about the origin on which f is analytically conjugate to $T_\lambda : x \to \lambda x$. We denote by $r(f)$ the exact radius of Δ_f and introduce a function $\tilde{r} = \tilde{r}(\lambda)$ that estimates $r(f)$ from below. There is an explicit formula for $\tilde{r}(\lambda)$, to be given in formula (11) in Section 4, with the properties stated in Theorem A and B below.

Theorem A. *Let $f \in \mathcal{G}_\lambda$, then $r(f) \geqslant \tilde{r}(\lambda)$.*

The proof is based on estimates of the coefficients of the conjugacy function H_f and an application of the Weierstrass Preparation Theorem. As our main result, refining the estimates of H_f for quadratic maps, we obtain the exact size of quadratic linearization disks.

Theorem B. *Let $p \geqslant 3$ and $\lambda \in \mathbb{C}_p$ be not a root of unity and put*

$$P_\lambda(x) := \lambda x + x^2 \in \mathbb{C}_p[x], \quad \text{where } 1/p < |1-\lambda| < 1.$$

Then, the linearization disk Δ_{P_λ} is the open disk $D_{r(P_\lambda)}(0)$ where $r(P_\lambda) = |1-\lambda|^{-1/p}\tilde{r}(\lambda)$.

In view of Theorem B, the general estimate $\tilde{r}(\lambda)$ is nearly optimal. The work lies in obtaining the optimal estimates of the coefficients of H_{P_λ} carried out in Sections 4 and 5. To our knowledge, this is the first known case in $\mathbb{C}_p$ where the exact linearization disk is known without having an explicit formula for the conjugacy H_f. In fact, the radius $r(P_\lambda)$ is invariant under power series perturbations.