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Finite multiplicity theorems for induction and restriction

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Abstract

We find upper and lower bounds of the multiplicities of irreducible admissible representations π of a semisimple Lie group G occurring in the induced representations $\mathrm{Ind}_H^G\,\tau$ from irreducible representations τ of a closed subgroup H. As corollaries, we establish geometric criteria for finiteness of the dimension of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\,\tau)$ (induction) and of $\mathrm{Hom}_H(\pi|_H, \tau)$ (restriction) by means of the real flag variety G/P, and discover that uniform boundedness property of these multiplicities is independent of real forms and characterized by means of the complex flag variety.

MSC: primary 22E46; secondary 14M17, 32A45, 35A27, 53C35

Keywords: Real reductive group; Admissible representation; Multiplicity; Hyperfunction; Unitary representation; Spherical variety; Symmetric space

Contents

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1. Introduction

The motivation of this work is the following fundamental questions in non-commutative harmonic analysis beyond symmetric spaces and branching problems of infinite-dimensional representations of real reductive Lie groups:

1. (Induction) What is the 'most general setting' of homogeneous spaces G/H in which we could expect reasonable and detailed analysis of function spaces on G/H?

2. (Restriction) What is the 'most general setting' of pairs (G, H) for which we could expect reasonable and detailed analysis of branching laws of the restriction of (arbitrary) irreducible representations of G to H?

We shall give an answer to these questions from the viewpoint of multiplicities of irreducible representations.

Let G be a connected real semisimple Lie group with finite center, and H a closed (not necessarily, reductive) subgroup with at most finitely many connected components. (It is easy to see that the results of this article remain true if we replace connected semisimple Lie groups G by linear reductive groups.)

We consider the following two geometric conditions:

$$\text{There exists an open } H\text{-orbit on the real flag variety } G/P. \tag{HP}$$

$$\text{There exists an open } H_c\text{-orbit on the complex flag variety } G_c/B. \tag{HB}$$

Here P is a minimal parabolic subgroup of G, B is a Borel subgroup of a complex Lie group G_c with the complexified Lie algebra $\mathfrak{g}_c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and H_c a complex subgroup with Lie algebra $\mathfrak{h}_c = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively. The condition (HB) is equivalent to that G_c/H_c is *spherical* (i.e. B has an open orbit on G_c/H_c) when $G \supset H$ are defined algebraically. Similarly, we call G/H is *real spherical* [14] if (HP) is satisfied (i.e. P has an open orbit on G/H), see Remark 2.5(4) for equivalent definitions.

An analogous notation $P_H \subset H$ and $B_H \subset H_c$ will be applied when H is reductive. In this case we can consider also the following two conditions:

$$\text{There exists an open } P_H\text{-orbit on the real flag variety } G/P. \tag{PP}$$

$$\text{There exists an open } B_H\text{-orbit on the complex flag variety } G_c/B. \tag{BB}$$

Clearly, these four conditions on the pair (G, H) do not depend on the choice of parabolics, coverings or connectedness of the groups, but are determined locally, namely, only by the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. An easy argument (see Lemma 4.2) shows that the following implications hold. Here we consider (PP) and (BB) when H is reductive: