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Normal cycles and curvature measures of sets with d.c. boundary ☆

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Abstract

We show that for every compact domain in a Euclidean space with d.c. (delta-convex) boundary there exists a unique Legendrian cycle such that the associated curvature measures fulfill a local version of the Gauss–Bonnet formula. This was known in dimensions two and three and was open in higher dimensions. In fact, we show this property for a larger class of sets including also lower-dimensional sets. We also describe the local index function of the Legendrian cycles and we show that the associated curvature measures fulfill the Crofton formula.

MSC: 26B25; 53C65

Keywords: D.c. function; Subgradient; Monge–Ampère function; Aura; Curvature measure; Legendrian cycle; Gauss–Bonnet formula

1. Introduction

The goal of extending the notion of curvature to non-smooth sets (with singularities) belongs to important tasks of geometry for decades. We consider here only subsets of the Euclidean

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space $\mathbb{R}^d$, though some approaches can be transferred to the Riemannian setting. It turned out that curvature measures can be derived from a more complex structure called normal cycle; this idea can be found by Sulanke and Wintgen [19] for smooth sets, Zähle [21] for sets with positive reach, Fu [5] for more general sets, and later developed by others.

To describe the basic idea, consider a full-dimensional compact subset A of $\mathbb{R}^d$ with C^2-smooth boundary, and let $\operatorname{nor} A$ be its unit normal bundle, i.e., $\operatorname{nor} A$ consists of pairs (x, n), where x is a boundary point of A and n is the unit outer normal vector to A at x. The normal cycle N_A of A is the $(d-1)$-dimensional current which is given by integrating over the oriented manifold $\operatorname{nor} A$, i.e.,

$$N_A(\phi) = \int\limits_{\operatorname{nor} A} \phi = \int\limits_{\operatorname{nor} A} \langle \xi_A, \phi \rangle \, d\mathcal{H}^{d-1}$$

for any smooth $(d-1)$-form ϕ on $\mathbb{R}^{2d}$ (here ξ_A is a prescribed unit simple $(d-1)$-vectorfield orienting $\operatorname{nor} A$ and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure).

Given $k \in \{0, \dots, d-1\}$, let φ_k be the kth Lipschitz–Killing differential $(d-1)$-form on $\mathbb{R}^{2d}$ which can be described by

$$\begin{aligned}&\big\langle a^1 \wedge \cdots \wedge a^{d-1}, \varphi_k(x,n) \big\rangle \\ &\quad = \mathcal{O}_{d-k-1}^{-1} \sum_{\sum_i \sigma(i) = d-1-k} \big\langle \pi_{\sigma(1)} a^1 \wedge \cdots \wedge \pi_{\sigma(d-1)} a^{d-1} \wedge n, \Omega_d \big\rangle,\end{aligned}$$

where a^i are vectors from $\mathbb{R}^{2d}$, $\pi_0(x,n) = x$ and $\pi_1(x,n) = n$ are coordinate projections, the sum is taken over finite sequences σ of values from $\{0, 1\}$, Ω_d denotes the volume form in $\mathbb{R}^d$ and $\mathcal{O}_{d-1} = \mathcal{H}^{d-1}(S^{d-1}) = 2\pi^{d/2}/\Gamma(\frac{d}{2})$. Note that, in particular, φ_0 is an $\mathcal{O}_{d-1}^{-1}$-multiple of the π_1-pull-back of the volume form $n \lrcorner \Omega_d$ on S^{d-1}:

$$\varphi_0 = \mathcal{O}_{d-1}^{-1} \pi_1^{\#}(n \lrcorner \Omega_d).$$

Integrating φ_k over $\operatorname{nor} A$ yields the kth (total) curvature of A, which can also be expressed as the integral of the kth symmetric function of principal curvatures of A:

$$N_A(\varphi_k) = \int\limits_{\operatorname{nor} A} \varphi_k = C_k(A).$$

For completeness, we define also $C_d(A) = \mathcal{H}^d(A)$. The kth curvature measure of A, $C_k(A, \cdot)$, is obtained by localizing with a Borel set $F \subset \mathbb{R}^d$

$$C_k(A, F) = \big(N_A \llcorner \big(F \times \mathbb{R}^d\big)\big)(\varphi_k) = \int\limits_{\operatorname{nor} A} (\mathbf{1}_F \circ \pi_0) \varphi_k.$$

In case of sets with singularities, the normal direction need not be determined uniquely. A useful and well tractable set class containing both smooth sets and closed convex sets is the family of sets with positive reach (i.e., sets for which any point within a certain distance apart has its unique nearest point in the set). Federer [3] introduced curvature measures for sets with positive reach by means of a local Steiner formula, and Zähle [21] defined normal cycles for these sets.

Fu [5] observed that the normal cycle of a set has a tangential property called later Legendrian, and he called *Legendrian cycle* any closed rectifiable $(d-1)$-dimensional current in $\mathbb{R}^d \times S^{d-1}$