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Pointed homotopy and pointed lax homotopy of 2-crossed module maps

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Abstract

We address the (pointed) homotopy theory of 2-crossed modules (of groups), which are known to faithfully represent Gray 3-groupoids, with a single object, and also connected homotopy 3-types. The homotopy relation between 2-crossed module maps will be defined in a similar way to Crans' 1-transfors between strict Gray functors, however being pointed, thus this corresponds to Baues' homotopy relation between quadratic module maps. Despite the fact that this homotopy relation between 2-crossed module morphisms is not, in general, an equivalence relation, we prove that if $\mathcal{A}$ and $\mathcal{A}'$ are 2-crossed modules, with the underlying group F of $\mathcal{A}$ being free (in short $\mathcal{A}$ is free up to order one), then homotopy between 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$ yields, in this case, an equivalence relation. Furthermore, if a chosen basis B is specified for F, then we can define a 2-groupoid $\mathrm{HOM}_B(\mathcal{A}, \mathcal{A}')$ of 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, homotopies connecting them, and 2-fold homotopies between homotopies, where the latter correspond to (pointed) Crans' 2-transfors between 1-transfors.

We define a partial resolution $Q^1(\mathcal{A})$, for a 2-crossed module $\mathcal{A}$, whose underlying group is free, with a canonical chosen basis, together with a projection map $\mathrm{proj} : Q^1(\mathcal{A}) \to \mathcal{A}$, defining isomorphisms at the level of 2-crossed module homotopy groups. This resolution (which is part of a comonad) leads to a weaker notion of homotopy (lax homotopy) between 2-crossed module maps, which we fully develop and describe. In particular, given 2-crossed modules $\mathcal{A}$ and $\mathcal{A}'$, there exists a 2-groupoid $\mathcal{HOM}_{\mathrm{LAX}}(\mathcal{A}, \mathcal{A}')$ of (strict) 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, and their lax homotopies and lax 2-fold homotopies, leading to the question of whether the category of 2-crossed modules and strict maps can be enriched over the monoidal category *Gray*.

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The associated notion of a (strict) 2-crossed module map $f : \mathcal{A} \to \mathcal{A}'$ to be a lax homotopy equivalence has the two-of-three property, and it is closed under retracts. This discussion leads to the issue of whether there exists a model category structure in the category of 2-crossed modules (and strict maps) where weak equivalences correspond to lax homotopy equivalences, and any free up to order one 2-crossed module is cofibrant.

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1. Introduction and simplicial group background/context

Let $\mathcal{G} = (G_n, d_i^n, s_i^n; i \in \{0, 1, \ldots, n\}, n = 0, 1, 2, \ldots)$ be a simplicial group; [45,26,38,18]. As usual, see for example [40,25], we say that $\mathcal{G}$ is free if each group G_n of n-simplices is a free group, with a chosen basis, and these basis are stable under the degeneracy maps $s_i^n : G_n \to G_{n+1}$. Recall that the Moore complex [44,43,16] $N(\mathcal{G})$ of a simplicial group $\mathcal{G}$ is given by the (normal) complex of groups $(\ldots \to A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_0 = G_0)$, where:

$$A_n = \bigcap_{i=0}^{n-1} \ker\left(d_i^n\right),$$

and $\partial_n : A_n \to A_{n-1}$ is the restriction of the boundary map $d_n^n : G_n \to G_{n-1}$. We say that the Moore complex of $\mathcal{G}$ has length n if the unique (possibly) non-trivial components of $N(\mathcal{G})$ are $A_{n-1} \to A_{n-2} \to \cdots \to A_0$. (Here "length" correspond to the number of groups, rather than the number of arrows, which is the usual convention). Not surprisingly, this Moore complex has a lot of extra structure, defining what a hyper crossed complex is [15], which retains enough information to recover the original simplicial group, up to isomorphism. This contains two well known results, stating that the categories of simplicial groups with Moore complexes of length two and three (respectively) are equivalent to the categories of crossed modules and of 2-crossed modules of groups (respectively), see [44,43,16], the latter being exactly hyper crossed complexes of length two and three (respectively); we will go back to this issue below. Hyper crossed complexes therefore generalise both crossed modules and 2-crossed modules.

Looking at the last two stages of the Moore complex $N(\mathcal{G})$ of a simplicial group $\mathcal{G}$, namely $\partial = \partial_1 : N_1(\mathcal{G}) \to N_0(\mathcal{G})$, one has an induced action of $N_0(\mathcal{G})$ on $N_1(\mathcal{G})$ by automorphisms, and also the action of $N_0(\mathcal{G})$ on itself by conjugation, and the boundary map $\partial : N_1(\mathcal{G}) \to N_0(\mathcal{G})$ preserves these actions; in other words one has a pre-crossed module ([9,3,4,40]), called the pre-crossed module associated to the simplicial group $\mathcal{G}$.

The homotopy groups of a simplicial group $\mathcal{G}$ are, by definition, given by the homology groups of its Moore complex $N(\mathcal{G})$ (which is a normal complex of, not necessarily abelian, groups). These correspond to the homotopy groups of the simplicial group $\mathcal{G}$ seen as a simplicial set (despite the fact that $\pi_0(S)$, for S a simplicial set, is not in general a group but a set). Note that simplicial groups are Kan complexes, and therefore their homotopy groups are well defined [38,18].

It is a fundamental result of Quillen [45,26] that the category of simplicial groups is a model category, where weak equivalences are the simplicial group maps $f : \mathcal{G} \to \mathcal{G}'$, inducing isomorphisms at the level of homotopy groups, and fibrations are the simplicial groups maps $f : \mathcal{G} \to \mathcal{G}'$