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ADVANCES IN Mathematics

Advances in Mathematics 240 (2013) 24-60

www.elsevier.com/locate/aim

# Plane posets, special posets, and permutations

L. Foissy

Laboratoire de Mathématiques, Université de Reims, Moulin de la Housse - BP 1039 - 51687 REIMS Cedex 2, France

Received 5 October 2012; accepted 5 March 2013 Available online 28 March 2013

Communicated by Michel Van den Bergh

## Abstract

We study the self-dual Hopf algebra  $\mathcal{H}_{S\mathcal{P}}$  of special posets introduced by Malvenuto and Reutenauer and the Hopf algebra morphism from  $\mathcal{H}_{S\mathcal{P}}$  to the Hopf algebra of free quasi-symmetric functions **FQSym** given by linear extensions. In particular, we construct two Hopf subalgebras both isomorphic to **FQSym**; the first one is based on plane posets, the second one on heap-ordered forests. An explicit isomorphism between these two Hopf subalgebras is also defined, with the help of two combinatorial transformations on special posets. The restriction of the Hopf pairing of  $\mathcal{H}_{S\mathcal{P}}$  to these Hopf subalgebras and others is also studied, as well as certain isometries between them. These problems are solved using duplicial and dendriform structures.

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MSC: 06A11; 05A05; 16W30; 17A30

Keywords: Special posets; Permutations; Self-dual Hopf algebras; Duplicial algebras; Dendriform algebras

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E-mail address: loic.foissy@univ-reims.fr.

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### 0. Introduction

The Hopf algebra of double posets is introduced in [17]. Recall that a *double poset* is a finite set with two partial orders; the set of isoclasses of double posets is given a structure of monoid, with a product called *composition* (Definition 4). The algebra of this monoid is given a coassociative coproduct, with the help of the notion of *ideal* of a double poset. We then obtain a graded, connected Hopf algebra, non commutative and non cocommutative. This Hopf algebra  $\mathcal{H}_{DP}$  is self-dual: it has a nondegenerate Hopf pairing  $\langle -, - \rangle$ , such that the pairing of two double posets is given by the number of *pictures* between these double posets (Definition 6); see [6] for more details on the nondegeneracy of this pairing.

Other algebraic structures are constructed on  $\mathcal{H}_{DP}$  in [6]. In particular, a second product is defined on  $\mathcal{H}_{DP}$ , making it a free 2-As Hopf algebra [13]. As a consequence, this object is closely related to operads and the theory of combinatorial Hopf algebras [14]. In particular, it contains the free 2-As algebra on one generator: this is the Hopf subalgebra  $\mathcal{H}_{WNP}$  of WN posets, see Definition 3. Another interesting Hopf subalgebra  $\mathcal{H}_{PP}$  is given by plane posets, that is to say double poset with a particular condition of (in)compatibility between the two orders (Definition 2).

We investigate in the present text the algebraic properties of the family of *special posets*, that is to say double posets such that the second order is total [17]. They generate a Hopf subalgebra of  $\mathcal{H}_{DP}$  denoted by  $\mathcal{H}_{SP}$ . For example, as explained in [6], the two partial orders of a plane poset allow to define a third, total order, so plane posets can also be considered as special posets: this defines an injective morphism of Hopf algebras from  $\mathcal{H}_{PP}$  to  $\mathcal{H}_{SP}$ . Its image is denoted by  $\mathcal{H}_{SPP}$ . Another interesting Hopf subalgebra of  $\mathcal{H}_{SP}$  is generated by the set of *ordered forests*; Download English Version:

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