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The boundary value problem for discrete analytic functions

M. Skopenkov*

Institute for Information Transmission Problems of the Russian Academy of Sciences, Bolshoy Karetny per. 19, bld. 1, Moscow, 127994, Russian Federation
King Abdullah University of Science and Technology, Saudi Arabia

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Abstract

This paper is on further development of discrete complex analysis introduced by R. Isaacs, J. Ferrand, R. Duffin, and C. Mercat. We consider a graph lying in the complex plane and having quadrilateral faces. A function on the vertices is called discrete analytic, if for each face the difference quotients along the two diagonals are equal.

We prove that the Dirichlet boundary value problem for the real part of a discrete analytic function has a unique solution. In the case when each face has orthogonal diagonals we prove that this solution uniformly converges to a harmonic function in the scaling limit. This solves a problem of S. Smirnov from 2010. This was proved earlier by R. Courant–K. Friedrichs–H. Lewy and L. Lusternik for square lattices, by D. Chelkak–S. Smirnov and implicitly by P.G. Ciarlet–P.-A. Raviart for rhombic lattices.

In particular, our result implies uniform convergence of the finite element method on Delaunay triangulations. This solves a problem of A. Bobenko from 2011. The methodology is based on energy estimates inspired by alternating-current network theory.

MSC: 39A12; 31C20; 65M60; 60J45

Keywords: Discrete analytic function; Boundary value problem; Energy; Alternating current

* Correspondence to: Institute for Information Transmission Problems of the Russian Academy of Sciences, Bolshoy Karetny per. 19, bld. 1, Moscow, 127994, Russian Federation.
E-mail address: skopenkov@rambler.ru.
URL: http://skopenkov.ru.

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1. Introduction

Various constructions of complex analysis on planar graphs were introduced by Isaacs, Ferrand, Duffin, Mercat [19,15,12,24,25], Dynnikov–Novikov [14], Bobenko–Mercat–Suris [3], and Bobenko–Pinkall–Springborn [4]. Recently this subject is developed extensively due to applications to statistical physics [30], numerical analysis [18,2], computer graphics [1,31], and combinatorial geometry [26]; see [22,30] for recent surveys.

This paper concerns *linear* complex analysis on quadrilateral lattices [3]. A *quadrilateral lattice* is a finite graph $Q \subset \mathbb{C}$ with rectilinear edges such that each bounded face is a quadrilateral (not necessarily convex). Depending on the shape of faces, one speaks about *square*, *rhombic*, or *orthogonal* lattices (the latter is quadrilateral lattices such that the diagonals of each face are orthogonal); see Fig. 1. Different types of lattices are required for different applications; see Section 5.

A complex-valued function f on the vertices of Q is called *discrete analytic* [25], if the difference quotients along the two diagonals of each face are equal, i.e.,

$$\frac{f(z_1) - f(z_3)}{z_1 - z_3} = \frac{f(z_2) - f(z_4)}{z_2 - z_4} \tag{1}$$

for each quadrilateral face $z_1z_2z_3z_4$; see Fig. 1 to the right. The motivation for this definition is that both sides of Eq. (1) approximate the derivative of an analytic function f inside this face. The real part of a discrete analytic function is called a *discrete harmonic function*.

Discrete complex analysis is analogous to the classical complex analysis in many aspects [22]. One of the most natural and at the same time challenging problems is to prove convergence of discrete theory to the continuous one when the lattice becomes finer and finer [30]. A natural formalization of such *convergence* is uniform convergence of the solution of the Dirichlet boundary value problem for a discrete harmonic function to a harmonic function in the scaling limit.

1.1. Previous work

Convergence in this sense was proved by R. Courant–K. Friedrichs–H. Lewy [11, Section 4] and L. Lusternik [23, Sections 4–5] for square lattices, by D. Chelkak–S. Smirnov [8, Proposition 3.3] and (implicitly and in less general setup) by P.G. Ciarlet–P.-A. Raviart [10, Theorem 2] for rhombic lattices. In fact convergence for rhombic lattices is equivalent to convergence of the classical finite element method [12]. The latter subject is well-developed; see a survey [7] and a textbook [6]. Nonrhombic lattices cannot be accessed by known methods. Weaker convergence results not involving boundary value problems were obtained in [24, Theorem 3], [18, Theorem 2].

1.2. Contributions

We prove that the Dirichlet boundary value problem for a discrete harmonic function on a quadrilateral lattice has a unique solution. Our main result is that in the case of orthogonal lattices this solution uniformly converges to a harmonic function in the scaling limit; see Convergence Theorem 1.2 below. This solves a problem of S. Smirnov [30, Question 1].