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Modular forms and K3 surfaces

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Abstract

For every known Hecke eigenform of weight 3 with rational eigenvalues we exhibit a K3 surface over $\mathbb{Q}$ associated to the form. This answers a question asked independently by Mazur and van Straten. The proof builds on a classification of CM forms by the second author.

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1. Introduction

The question of modularity for algebraic varieties over $\mathbb{Q}$ has been studied in great detail in recent years. Historically, it began with work by A. Weil on Fermat varieties [44], continued in the context of curves by Deuring [7] and Eichler [9]. Shimura then proved that every Hecke eigenform of weight 2 is associated to an abelian variety over $\mathbb{Q}$ (conf. [35, Section 7]). In the case of rational eigenvalues, the corresponding variety is an elliptic curve.

Conversely, the Taniyama–Shimura–Weil conjecture states that every elliptic curve over $\mathbb{Q}$ is modular. The celebrated proof of this conjecture by Wiles et al. [46,43,4] not only implies Fermat's Last Theorem, but also catalyzed many further developments in this area, notably the proof of Serre's conjecture by Khare and Wintenberger [21]. This implies modularity for several classes of varieties, for instance rigid Calabi–Yau threefolds over $\mathbb{Q}$ (cf. [8,16]). These results

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were preceded by work by Livné [22] on modularity for two-dimensional motives with complex multiplication (CM) that we use here, citing it as Theorem 2.

On the other hand, the problem of geometric realizations is harder for Hecke eigenforms of weight greater than two. Deligne [6] gives a geometric construction of ℓ-adic Galois representations for Hecke eigenforms. However, the varieties involved vary greatly with the level. In this sense, Deligne's construction is not as uniform as one might wish (cf. Remark 2).

This paper solves the first case of higher weight where we can realize *all known Hecke eigenforms* with rational eigenvalues in a single class of varieties. Conjecturally these comprise all Hecke eigenforms in question, as stated in our main theorem:

Theorem 1. *Assume the extended Riemann Hypothesis (ERH) for odd real Dirichlet characters. Then every Hecke eigenform of weight* 3 *with rational eigenvalues is associated to a* K3 *surface over* $\mathbb{Q}$.

This result answers a question asked independently by Mazur and van Straten. It builds on the classification of CM forms with rational coefficients by the second author which we recall in Section 3. That section also explains the dependence of Theorem 1 on the ERH. Section 2 recalls the notion of singular K3 surfaces and Livné's modularity result (Theorem 2). We review the relevant known examples and obstructions in Sections 4 and 5. For every known Hecke eigenform of weight 3 with rational eigenvalues we then exhibit an explicit singular K3 surface over $\mathbb{Q}$, thus proving Theorem 1. Our main technique to achieve this is constructing one-dimensional families of K3 surfaces and searching for singular specializations over $\mathbb{Q}$. This is explained in Section 6 and exhibited in detail for one particular family in Section 7. The paper concludes with the remaining surfaces needed to prove Theorem 1. A summary of the proof is given in Section 9.

2. Singular K3 surfaces

A K3 surface is a smooth, projective, simply connected surface X with trivial canonical bundle $\omega_X = \mathcal{O}_X$. The most prominent examples are smooth quartics in $\mathbb{P}^3$ and Kummer surfaces. Later we will work with elliptic K3 surfaces.

Throughout this paper, modularity will refer to classical modular forms (cf. Section 3). This classical kind of modularity is a very special property of a variety; a general K3 surface over $\mathbb{Q}$ cannot be modular for several reasons (cf. the discussion before Theorem 2), though the Langlands Program predicts a correspondence with some automorphic forms.

K3 surfaces and their moduli have been studied in great detail. We will come back to these questions in Section 6. The only complex K3 surfaces that can be classically modular are those that have no moduli at all. In terms of the Picard number $\rho(X) = \operatorname{rk} \operatorname{NS}(X)$, the condition that X have no moduli is that

$$\rho(X) = 20,$$

the maximum in characteristic zero. K3 surfaces with Picard number 20 are often referred to as *singular K3 surfaces*. The terminology is reflected in the Shioda–Inose structure (cf. Section 4) which relates any singular K3 surface to a product of two isogenous elliptic curves with complex multiplication (CM), thus with *singular moduli*.

Our results will often be stated in terms of the *discriminant* $d = d(X)$ of a singular K3 surface X, i.e. the discriminant of the intersection form on the Néron–Severi lattice, which is the Néron–Severi group endowed with the cup-product pairing:

$$d = d(X) = \operatorname{disc}(\operatorname{NS}(X)).$$