

Curvatures of homogeneous Randers spaces

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Abstract

We study curvatures of homogeneous Randers spaces. After deducing the coordinate-free formulas of the flag curvature and Ricci scalar of homogeneous Randers spaces, we give several applications. We first present a direct proof of the fact that a homogeneous Randers space is Ricci quadratic if and only if it is a Berwald space. We then prove that any left invariant Randers metric on a non-commutative nilpotent Lie group must have three flags whose flag curvature is positive, negative and zero, respectively. This generalizes a result of J.A. Wolf on Riemannian metrics. We prove a conjecture of J. Milnor on the characterization of central elements of a real Lie algebra, in a more generalized sense. Finally, we study homogeneous Finsler spaces of positive flag curvature and particularly prove that the only compact connected simply connected Lie group admitting a left invariant Finsler metric with positive flag curvature is $SU(2)$.

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1. Introduction

The main goal of this paper is to study curvatures of homogeneous Randers spaces. Our motivation is to classify homogeneous Finsler spaces of positive flag curvature. The study of positively curved manifold has been a classical and central problem in Riemannian geometry. However, till now this has only been successful in the homogeneous case. A sufficient and necessary condition for the existence of a positively curved Riemannian metric on a general manifold is still unknown, although a series of inhomogeneous Riemannian manifolds with positive sectional curvature have been constructed (see for example [2,16–19]). Nevertheless, in the homogeneous case, the problem is completely understood. In fact, by the work of M. Berger [5], Berard Bergery [3], Wallach [33] and Aloff–Wallach [1], a complete classification of (connected and simply connected) homogeneous manifolds admitting invariant Riemannian metric were achieved. More recently, Verdianni–Ziller [32] determined all the homogeneous Riemannian metrics on spheres which have positive sectional curvature.

In Finsler geometry this problem possesses the same importance. However, up to now only very little attention has been paid to this subject. In [20], the authors initiated the study of homogeneous Finsler spaces of positive flag curvature. The main result is a complete classification of homogeneous Randers spaces with isotropic S-curvature and positive flag curvature. It is natural to consider the same problem without the restriction in S-curvature. However, since an explicit formula of the flag curvature of homogeneous Randers spaces has not been obtained, this problem is still open.

In this paper we will continue the study of this important problem. The first part is devoted to the deduction of an explicit formula of the flag curvature of homogeneous Randers spaces as well as a formula for the Ricci scalar, without involving local coordinate systems. We then apply this formula to obtain several interesting results. We first give a direct proof of the fact that a homogeneous Randers space is Ricci quadratic if and only if it is a Berwald space. Then we generalize a result of J.A. Wolf [34] on the sectional curvature of left invariant Riemannian metrics on nilpotent Lie groups to the Randers case, namely, we prove that any left invariant

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