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Double MV cycles and the Naito–Sagaki–Saito crystal

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Abstract

The theory of MV cycles associated to a complex reductive group G has proven to be a rich source of structures related to representation theory. We investigate double MV cycles, which are analogues of MV cycles in the case of an affine Kac–Moody group. We prove an explicit formula for the Braverman–Finkelberg–Gaitsgory (2006) [7] crystal structure on double MV cycles, generalizing a finite-dimensional result of Baumann and Gaussent (2008) [2]. As an application, we give a geometric construction of the Naito–Sagaki–Saito [23] crystal via the action of $\widehat{SL}_n$ on Fermionic Fock space. In particular, this construction gives rise to an isomorphism of crystals between the set of double MV cycles and the Naito–Sagaki–Saito crystal. As a result, we can independently prove that the Naito–Sagaki–Saito crystal is the $B(\infty)$ crystal. In particular, our geometric proof works in the previously unknown case of $\widehat{\mathfrak{sl}}_2$.

Keywords: MV cycles; Affine Kac–Moody groups; Crystals; Fock space

1. Introduction

1.1. Crystals

Let $\mathfrak{g}$ be a Kac–Moody Lie algebra. Kashiwara invented the notion of a $\mathfrak{g}$-*crystal*, which is a combinatorial analogue of a $\mathfrak{g}$ representation. A crystal consists of a set $\mathbf{B}$ along with crystal operators and some auxiliary data (we will review the precise definition in Section 2.7). A rich source of crystals comes from actual representations: given an integrable representation in the BGG category $\mathcal{O}$ or a Verma module, we can canonically extract a crystal using Kashiwara's method of *crystal bases*.

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Let us give labels to two crystals that will be of interest to us. Using crystal bases, define $B(\lambda)$ to be the crystal associated to the irreducible representation $V(\lambda)$ of highest weight λ, and define $B(\infty)$ to be the crystal associated to the Verma module of highest weight zero (equivalently, the negative part of the universal enveloping algebra). By general theory of crystals, there is a recipe for recovering the various $B(\lambda)$ crystals from the crystal $B(\infty)$, and vice versa. Because of this equivalence, we will focus on the $B(\infty)$ crystal in our discussion.

An interesting problem is to find algebraic, geometric, and combinatorial realizations of the crystal $B(\infty)$. In the next subsection, we will review an algebraic realization of the crystal coming from Lusztig's canonical basis. In the subsequent subsection, we will review a geometric realization of the crystal coming from MV cycles in the affine Grassmannian. In the course of the discussion, we will explain how both realizations give rise to the same combinatorial realization of the $B(\infty)$ crystal via MV polytopes.

1.2. Lusztig's canonical basis

One particularly nice realization of the $B(\infty)$ crystal comes from Lusztig's canonical basis [20], which is a basis in the negative part of the quantum group enjoying many nice properties. When $\mathfrak{g}$ is finite type, there is a natural parameterization of the canonical basis coming from the various Poincaré–Birkhoff–Witt (PBW) bases. An interesting question is to study the combinatorics that records how to pass between the various parameterizations of the canonical basis coming from the various PBW bases. Lusztig gave an explicit answer in simply-laced cases [21]. Berenstein and Zelevinsky [6] gave an answer in all types. In addition, they indicate how the reparameterization data can be arranged into a combinatorial gadget called an *MV polytope* (we will explain this name, due to Kamnitzer, in the next subsection). In particular, they produce a bijection between the canonical basis and the set of MV polytopes. Finally, they give a combinatorial description of the crystal structure using only the data coming from MV polytopes.

1.3. MV cycles in the finite dimensional case

Let G be a complex reductive group. The geometric Satake equivalence of Lusztig [19], Beilinson–Drinfeld [5], Ginzburg [12], and Mirković–Vilonen [22] relates the geometry of the affine Grassmannian with the representation theory of the dual group $G^\vee$. The most recent proof, due to Mirković–Vilonen, provided even finer information; they gave an explicit basis for each irreducible representation of $G^\vee$ indexed by certain irreducible subvarieties of the affine Grassmannian of G. These irreducible subvarieties are called *Mirković–Vilonen (MV) cycles*, and they are highly structured.

Remark. MV cycles come in two flavors: there are those that correspond to basis vectors in irreducible representations, and there are those that correspond to basis vectors in Verma modules. In this paper we will focus almost exclusively on MV cycles corresponding to basis vectors in Verma modules. Unless we specify otherwise, we will mean this latter variety when we write "MV cycles".

Let us highlight some key results in the theory of MV cycles, which will be relevant to our later discussion:

- Braverman, Finkelberg, and Gaitsgory [9,7] proved that the MV cycles corresponding to basis vectors in irreducible representations carry a natural crystal structure for the dual Lie algebra. More specifically, corresponding to each irreducible representation $V(\lambda)$ of $G^\vee$ with highest