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Uniform hypergraphs containing no grids

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Abstract

A hypergraph is called an $r \times r$ *grid* if it is isomorphic to a pattern of r horizontal and r vertical lines, i.e., a family of sets $\{A_1, \ldots, A_r, B_1, \ldots, B_r\}$ such that $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $1 \leq i < j \leq r$ and $|A_i \cap B_j| = 1$ for $1 \leq i, j \leq r$. Three sets C_1, C_2, C_3 form a *triangle* if they pairwise intersect in three distinct singletons, $|C_1 \cap C_2| = |C_2 \cap C_3| = |C_3 \cap C_1| = 1$, $C_1 \cap C_2 \neq C_1 \cap C_3$. A hypergraph is *linear*, if $|E \cap F| \leq 1$ holds for every pair of edges $E \neq F$.

In this paper we construct large linear r-hypergraphs which contain no grids. Moreover, a similar construction gives large linear r-hypergraphs which contain neither grids nor triangles. For $r \geq 4$ our constructions are almost optimal. These investigations are motivated by coding theory: we get new bounds for optimal superimposed codes and designs.

MSC: 05D05; 11B25

Keywords: Turán hypergraph problem; Density problems; Union free hypergraphs; Superimposed codes

1. Sparse hypergraphs, designs, and codes

In this section we first present some previous investigations in extremal set theory on the topic described in the abstract. Then we state our main theorem. This is followed by motivations

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in coding theory and corollaries where we improve the previously known bounds for so called optimal superimposed codes and designs. To prove the main theorem we are using tools from combinatorial number theory and discrete geometry given in Section 2. In Section 3 we present constructions proving the stated theorems, followed by remarks on union-free and cover-free graphs and triple systems.

1.1. Avoiding grids in linear hypergraphs

Speaking about a hypergraph $\mathbb{F} = (V, \mathcal{F})$ we frequently identify the vertex set $V = V(\mathbb{F})$ by the set of first integers $[n] := \{1, 2, \ldots, n\}$, or points on the plane $\mathbf{R}^2$, or elements of a q-element finite field F_q. To shorten notations we frequently say 'hypergraph $\mathcal{F}$' (or set system $\mathcal{F}$) thus identifying $\mathbb{F}$ to its edge set $\mathcal{F}$. $\mathbb{F}$ is *linear* if for all $A, B \in \mathcal{F}$, $A \neq B$ we have $|A \cap B| \leq 1$. The *degree*, $\deg_{\mathbb{F}}(x)$, of an element $x \in [n]$ is the number of hyperedges in $\mathcal{F}$ containing x. $\mathbb{F}$ is *regular* if every element $x \in [n]$ has the same degree. It is *uniform* if every edge has the same number of elements, r-uniform means $|F| = r$ for all $F \in \mathcal{F}$. An $(n, r, 2)$-*packing* is a linear r-uniform hypergraph $\mathcal{P}$ on n vertices. Obviously, $|\mathcal{P}| \leq \binom{n}{2} / \binom{r}{2}$. If here equality holds, then $\mathcal{P}$ is called an $S(n, r, 2)$ *Steiner system*.

Definition 1.1. A set system $\mathcal{F}$ contains an $a \times b$ grid if there exist two disjoint subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ such that

- $|\mathcal{A}| = a$, $|\mathcal{B}| = b$, $|\mathcal{A} \cup \mathcal{B}| = a + b$,
- $A \cap A' = B \cap B' = \emptyset$ for all $A, A' \in \mathcal{A}$, $A \neq A'$, $B, B' \in \mathcal{B}$, $B \neq B'$, and
- $|A \cap B| = 1$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Thus an r-uniform $r \times r$ grid, $\mathbb{G}_{r\times r}$, is a disjoint pair $\mathcal{A}, \mathcal{B}$ of the same sizes r such that they cover exactly the same set of r^2 elements.

Theorem 1.2. *For $r \geq 4$ there exists a real $c_r > 0$ such that there are linear r-uniform hypergraphs $\mathcal{F}$ on n vertices containing no grids and*

$$|\mathcal{F}| > \frac{n(n-1)}{r(r-1)} - c_r n^{8/5}.$$

The proof is postponed to Section 3.2.

The *Turán number* of the r-uniform hypergraph $\mathcal{H}$, denoted by $\mathrm{ex}(n, \mathcal{H})$, is the size of the largest $\mathcal{H}$-free r-graph on n vertices. If we want to emphasize r, then we write $\mathrm{ex}_r(n, \mathcal{H})$. Let $\mathbb{I}_{\geq 2}$ be (more precisely $\mathbb{I}^r_{\geq 2}$) the class of r-graphs of two edges and intersection sizes at least two. This class consists of $r - 2$ non-isomorphic hypergraphs, $\mathcal{I}_j$, $2 \leq j < r$, $\mathcal{I}_j := \{A_j, B_j\}$ such that $|A_j| = |B_j| = r$, $|A_j \cap B_j| = j$. Using these notations the above theorem can be restated as follows.

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \mathrm{ex}_r(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r\times r}\}) \leq \frac{n(n-1)}{r(r-1)} \tag{1}$$

holds for every $n, r \geq 4$. In the case of $r = 3$ we only have

$$\Omega(n^{1.8}) = \mathrm{ex}_3(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{3\times 3}\}) \leq \frac{1}{6} n(n-1), \tag{2}$$

see in Section 1.5. The case of graphs, $r = 2$, is different, see later in Section 3.5.