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Hadamard multipliers on spaces of real analytic functions

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Abstract

We consider multipliers on spaces of real analytic functions of one variable, i.e., maps for which monomials are eigenvectors. We characterize sequences of complex numbers which are sequences of eigenvalues for some multiplier. We characterize invertible multipliers, in particular, we find which Euler differential operators of infinite order have global analytic solutions on the real line. We present a number of examples where our theory applies. In some cases we give algorithms for solving the respective equations. Perturbation results for solvability are presented.
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1. Introduction

The aim of this paper is to find criteria of global analytic solvability in f of an Euler differential equation:

$$\sum_{n=0}^{\infty} a_n \theta^n f(t) = g(t), \quad t \in \mathbb{R}, \qquad (a_n)_{n\in\mathbb{N}} \subset \mathbb{C} \tag{1}$$

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where $\theta(h)(t) = th'(t)$ or a generalized q-difference functional equation

$$\sum_{n=0}^{k} a_n f(q_n t) = g(t), \quad t, q_n \in \mathbb{R},\ a_n \in \mathbb{C} \tag{2}$$

with g a given analytic function. The main theme behind is the notion of a (Taylor) multiplier. We call a linear continuous operator $M : \mathscr{A}(I) \to \mathscr{A}(I)$ on the space of real analytic functions $\mathscr{A}(I)$, $I \subset \mathbb{R}$ open, to be a *multiplier* whenever every monomial is an eigenvector. The corresponding sequence of eigenvalues $(m_n)_{n\in\mathbb{N}}$ is called the *multiplier sequence* for M. Since polynomials are dense in $\mathscr{A}(I)$ the sequence $(m_n)_{n\in\mathbb{N}}$ uniquely determines M but monomials do not form a Schauder basis of $\mathscr{A}(I)$ (in fact, this space has no Schauder basis at all [18]) therefore M is not just a diagonal operator. This makes the whole theory complicated and interesting.

Apart form Euler differential operators and generalized q-difference operators (defined as left hand side formulae in (1) and (2)) there are plenty of other interesting examples of multipliers: for instance, various integral operators related to the Hardy averaging operator or Hadamard multipliers. See also [15] or [16] and the literature listed there.

The main problem is just a question of surjectivity of a suitable multiplier $M : \mathscr{A}(I) \to \mathscr{A}(I)$. We concentrate on the case where $I \subset \mathbb{R}$ is open connected (we say shortly *open interval*, for instance $I = \mathbb{R}$). Surjectivity of multipliers for $0 \notin I$ was completely characterized in [15]. Now, we consider the challenging case of $0 \in I$. Here surjectivity implies invertibility. Again, invertibility for $0 \notin I$ was fully characterized in [16] but for $0 \in I$ the situation is much more complicated. The case is challenging since, for instance, the Euler differential operators are singular. Moreover, if $\sum_{n=0}^{\infty} f_n z^n$ is the Taylor series of $f \in \mathscr{A}(I)$ at zero then $\sum_{n=0}^{\infty} f_n m_n z^n$ is the Taylor series of $M(f)$ at zero (see Proposition 2.1) which fully justifies the name multiplier.

In the complex case, i.e., when the operator acts on all holomorphic functions on a given domain containing zero, the corresponding operator is called the Hadamard multiplier [35,36]. Our case is different since the space of real analytic functions consists of *germs* of holomorphic functions.

In the present paper we completely describe multiplier sequences $(m_n)_{n\in\mathbb{N}}$ both in the "matricial language" (Corollary 3.1) and via interpolation properties of holomorphic functions with a restricted growth (Theorem 4.5, Corollary 4.6). The notion of a Mellin function or a Mellin pair plays the crucial role (see Definition 4.7). This allows to characterize completely invertible multipliers (Corollaries 5.10 and 5.11). The result becomes striking in case of Euler differential operators (Corollary 6.1) and we get that many perturbations of an invertible Euler differential operators are also invertible (Corollary 6.5). A characterization of solvability of (2) is given in Corollary 7.5.

The general results are of existential flavour. We provide also algorithmic methods for solving quite general Euler differential equations (see Theorem 6.6) and for q-difference equations (see Corollary 7.10). Examples from Section 7 also provide methods for constructing multipliers with a given multiplier sequence (see especially Theorems 7.7 and 7.13) and inverses (comp. Example 7.15).

The main tools used in the paper are the representation theorems from [15]. The paper owes much to the ideas of Brück and Müller [8] and to the continuation of analyticity results of Arakelyan [2]. The whole theory is developed for $\mathscr{A}(\mathbb{R})$ but then also for $\mathscr{A}(I)$ where $I \subset \mathbb{R}$ is a general open connected set containing zero.

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