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Advances in Mathematics 241 (2013) 1–17

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Eta-quotients and theta functions

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Received 29 October 2010; accepted 24 March 2013
Available online 9 April 2013

Communicated by George Andrews

Abstract

The Jacobi Triple Product Identity gives a closed form for many infinite product generating functions that arise naturally in combinatorics and number theory. Of particular interest is its application to Dedekind's eta-function $\eta(z)$, defined via an infinite product, giving it as a certain kind of infinite sum known as a theta function. Using the theory of modular forms, we classify all eta-quotients that are theta functions.

Keywords: Modular forms; q-series; Eta-quotients; Theta functions

1. Introduction and statement of results

The Jacobi Triple Product Identity states that

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{n=-\infty}^{\infty} z^{2m}x^{m^2}, \tag{1.1}$$

which is surprising because it gives a striking closed form expression for an infinite product. Using (1.1), one can derive many elegant q-series identities. For example, one has the Euler identity

$$q\prod_{n=1}^{\infty}(1-q^{24n}) = \sum_{k=-\infty}^{\infty}(-1)^k q^{(6k+1)^2} \tag{1.2}$$

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0001-8708/$ - see front matter
http://dx.doi.org/10.1016/j.aim.2013.03.019

and the Jacobi identity

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2} = \sum_{k=-\infty}^{\infty} q^{k^2}. \tag{1.3}$$

Both (1.2) and (1.3) can be viewed as identities involving Dedekind's eta-function $\eta(z)$, which is defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \tag{1.4}$$

where $q := e^{2\pi i z}$. It is well known that $\eta(z)$ is essentially a half-integral weight modular form, a fact which Dummit, Kisilevsky, and McKay [5] exploited to classify all the eta-products (functions of the form $\prod_{i=1}^{s} \eta(n_i z)^{t_i}$, where each n_i and each t_i is a positive integer) whose q-series have multiplicative coefficients. Martin [7] later obtained the complete list of integer weight eta-quotients (permitting the t_i to be negative) with multiplicative coefficients.

The right hand sides of both (1.2) and (1.3) also have an interpretation in terms of half-integral weight modular forms: they are examples of *theta functions*. Given a Dirichlet character ψ, the theta function $\theta_\psi(z)$ of ψ is given by

$$\theta_\psi(z) := \sum_n \psi(n) n^\delta q^{n^2}, \tag{1.5}$$

where $\delta = 0$ or 1 according to whether ψ is even or odd. The summation over n in (1.5) is over the positive integers, unless ψ is the trivial character, in which case the summation is over all integers. With this language, (1.2) becomes

$$\eta(24z) = \theta_{\chi_{12}}(z),$$

where $\chi_{12}(n) = \left(\frac{12}{n}\right)$ and $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol. This fact is subsumed into the theorem of Dummit, Kisilevsky, and McKay, as $\eta(24z)$ is an eta-product and any theta function necessarily has multiplicative coefficients. However, we note that (1.3) is equivalent to

$$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \theta_1(z),$$

which is covered neither by the theorem of Dummit, Kisilevsky, and McKay (as the left-hand side is a quotient of eta-functions, not merely a product), nor is it covered by the theorem of Martin (as the modular forms involved are of half-integral weight). It is therefore natural to ask which eta-quotients are theta functions.

Theorem 1.1. 1. *The following eta-quotients are the only ones which are theta functions for an even character:*

$$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\frac{\eta(8z)\eta(32z)}{\eta(16z)} = \sum_{n=1}^{\infty} \left(\frac{2}{n}\right) q^{n^2},$$