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Star clusters in independence complexes of graphs

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Abstract

We introduce the notion of *star cluster* of a simplex in a simplicial complex. This concept provides a general tool to study the topology of independence complexes of graphs. We use star clusters to answer a question arisen from works of Engström and Jonsson on the homotopy type of independence complexes of triangle-free graphs and to investigate a large number of examples which appear in the literature. We present an alternative way to study the chromatic and clique numbers of a graph from a homotopical point of view and obtain new results regarding the connectivity of independence complexes.

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Keywords: Independence complexes; Graphs; Simplicial complexes; Homotopy types; Homotopy invariants

1. Introduction

Since Lovász' proof of the Kneser conjecture in 1978, numerous applications of algebraic topology to combinatorics, and in particular to graph theory, have been found. A recurrent strategy in topological combinatorics consists of the study of homotopy invariants of certain CW-complexes constructed from a discrete structure to obtain combinatorial information about the original object. In Lovász' prototypical example, connectivity properties of the *neighborhood complex* $\mathcal{N}(G)$ of a graph G are shown to be closely related to the chromatic number $\chi(G)$ of G. Lovász conjecture states that there exists a similar relationship between the so called *Hom complexes* $\mathrm{Hom}(H, G)$ and $\chi(G)$ when H is a cycle with an odd number of vertices. The Hom

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complex $\mathrm{Hom}(H, G)$ is homotopy equivalent to $\mathcal{N}(G)$ when H is the complete graph on two vertices K_2. Babson and Kozlov [2] proved this conjecture in 2007. In their proof they used that $\mathrm{Hom}(G, K_n)$ is linked to another polyhedron associated to G, which is called the *independence complex* of G. Given a graph G, its independence complex I_G is the simplicial complex whose simplices are the independent sets of vertices of G. In this approach to the conjecture it was then needed to understand the topology of independence complexes of cycles. Independence complexes have also been used to study Tverberg graphs [21] and independent systems of representatives [1].

For any finite simplicial complex K there exists a graph G such that I_G is homeomorphic to K. Specifically, given a complex K, we consider the graph G whose vertices are the simplices of K and whose edges are the pairs (σ, τ) of simplices such that σ is not a face of τ and τ is not a face of σ. Then I_G is isomorphic to the barycentric subdivision K' of K. In particular, the homotopy types of independence complexes of graphs coincide with homotopy types of compact polyhedra.

In the last years a lot of attention has been drawn to study the general problem of determining all the possible homotopy types of the independence complexes of graphs in some particular class. For instance, Kozlov [25] investigates the homotopy types of independence complexes of cycles and paths, Ehrenborg and Hetyei [17] consider this question for forests, Engström [18] for claw-free graphs, Bousquet-Mélou, Linusson and Nevo [10] for some square grids, Braun [11] for Stable Kneser graphs and Jonsson [24] for bipartite graphs. Other results investigate how the topology of the independence complex changes when the graph is modified in some particular way. Engström [19] analyzes what happens when some special points of the graph are removed and Csorba [15] studies how subdivisions of the edges of a graph affect the associated complex.

The purpose of this paper is two-fold: to introduce a notion that allows the development of several techniques which are useful to study homotopy types of independence complexes, and to establish new relationships between combinatorial properties of graphs and homotopy invariants of their independence complexes. We have mentioned that independence complexes are closely related to Hom complexes and therefore, they can be used to study chromatic properties of graphs. In this paper we show that there is a direct connection between independence complexes, colorability of graphs and other related graph invariants. We will obtain lower bounds for the chromatic number of a graph in terms of a numerical homotopy invariant associated to its independence complex. On the other hand we will introduce some ideas that are used to study the connectivity of I_G in terms of combinatorial properties of G.

One of the motivating questions of this work appears in Engström's Thesis [20] and concerns the existence of torsion in the homology groups of independence complexes of triangle-free graphs (i.e. graphs which do not contain triangles). Recently, Jonsson [24] proved that for any finitely generated abelian group Γ and any integer $n \geq 2$, there exists a triangle-free graph G such that the (integral) homology group $H_n(I_G)$ is isomorphic to Γ. In fact, he shows that the homotopy types of independence complexes of bipartite graphs are exactly the same as the homotopy types of suspensions of compact polyhedra. This result had also been proved independently by Nagel and Reiner [28, Proposition 6.2]. Two natural questions arise from Jonsson's work. Can $H_1(I_G)$ have torsion for some triangle-free graph G? And furthermore, what are the homotopy types of independence complexes of triangle-free graphs? In order to give a solution to these problems we introduce the notion of *star cluster* of a simplex in a simplicial complex. The star cluster $SC(\sigma)$ of a simplex $\sigma \in K$ is just the union of the simplicial stars of the vertices of σ. In general these subcomplexes can have non-trivial homotopy type but we will see that if K is the independence complex of a graph, then the star cluster of every simplex is contractible (Lemma 3.2). These fundamental blocks are used to answer both questions stated above. We