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Advances in Mathematics 241 (2013) 58–78
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Maximal Blaschke products☆

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Received 5 October 2011; accepted 25 March 2013
Available online 17 April 2013

Communicated by Nikolai Makarov

Abstract

We consider the classical problem of maximizing the derivative at a fixed point over the set of all bounded analytic functions in the unit disk with prescribed critical points. We show that the extremal function is essentially unique and always an indestructible Blaschke product. This result extends the Nehari–Schwarz Lemma and leads to a new class of Blaschke products called maximal Blaschke products. We establish a number of properties of maximal Blaschke products, which indicate that maximal Blaschke products constitute an appropriate infinite generalization of the class of finite Blaschke products.

MSC: 30H05; 30J10; 35J60; 30H20; 30F45; 53A30

Keywords: Bounded analytic functions; Schwarz lemma; Extremal problems; Bergman spaces; Blaschke products

1. Introduction and results

Let H^∞ denote the space of all functions analytic and bounded in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ equipped with the norm

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty, \quad f \in H^\infty.$$

A sequence $\mathcal{C} = (z_j)$ in $\mathbb{D}$ is called an H^∞ critical set, if there exists a nonconstant function f in H^∞ whose critical points are precisely the points on the sequence $\mathcal{C}$ counting multiplicities.

☆ This research has been supported by the Deutsche Forschungsgemeinschaft (Grants: Ro 3462/3–1 and Ro 3462/3–2).
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http://dx.doi.org/10.1016/j.aim.2013.03.017

This means that if the point z_j occurs m times in the sequence, then f' has a zero at z_j of precise order m, and $f'(z) \neq 0$ for every $z \in \mathbb{D} \setminus \mathcal{C}$. For an H^∞ critical set $\mathcal{C}$ we define the subspace

$$\mathcal{F}_{\mathcal{C}} := \left\{ f \in H^\infty : f'(z) = 0 \text{ for any } z \in \mathcal{C} \right\}$$

of all functions $f \in H^\infty$ such that any point of the sequence $\mathcal{C}$ is a critical point of f (with at least the prescribed multiplicity).

Our first theorem shows that the set $\mathcal{F}_{\mathcal{C}}$ always contains a Blaschke product whose critical set is precisely the sequence $\mathcal{C}$. In fact, more is true:

Theorem 1.1. *Let $\mathcal{C} = (z_j)$ be an H^∞ critical set and let N denote the number of times that* 0 *appears in the sequence $\mathcal{C}$. Then the extremal problem*

$$\max\left\{ \operatorname{Re} f^{(N+1)}(0) : f \in \mathcal{F}_{\mathcal{C}},\ \|f\|_\infty \leq 1 \right\} \tag{1.1}$$

has a unique solution $B_{\mathcal{C}} \in \mathcal{F}_{\mathcal{C}}$. The extremal function $B_{\mathcal{C}}$ is an indestructible Blaschke product with critical set $\mathcal{C}$ and is normalized by $B_{\mathcal{C}}(0) = 0$ and $B_{\mathcal{C}}^{(N+1)}(0) > 0$. If $\mathcal{C}$ is a finite sequence consisting of m points, then $B_{\mathcal{C}}$ is a finite Blaschke product of degree $m+1$; otherwise, $B_{\mathcal{C}}$ is an infinite Blaschke product.

Note that Theorem 1.1 says that the critical points of the extremal function $B_{\mathcal{C}}$ are *exactly* the points of the sequence $\mathcal{C}$ with prescribed multiplicity, so there are no "extra critical points" and $\mathcal{C}$ is the critical set of $B_{\mathcal{C}}$.

The crucial part of Theorem 1.1 is the assertion that the extremal function $B_{\mathcal{C}}$ is always an *indestructible Blaschke product*. Recall that a Blaschke product is called indestructible (see [28,35]) if for any conformal automorphism T of the unit disk the composition $T \circ B_{\mathcal{C}}$ is again a Blaschke product. Note that postcomposition by such a conformal automorphism does not change the critical set of a function in H^∞. Therefore, for any conformal automorphism T of $\mathbb{D}$, we call the Blaschke product $T \circ B_{\mathcal{C}}$ a *maximal Blaschke product with critical set $\mathcal{C}$*.

If $N = 0$, then the extremal problem (1.1) is exactly the problem of maximizing the derivative at a point, i.e., exactly the character of Schwarz' lemma. Let us put this observation in perspective.

Remark 1.2 (*The Nehari–Schwarz Lemma*)**.** In the special case where $\mathcal{C}$ is a *finite* sequence, Theorem 1.1 is essentially the classical and well-known Nehari–Schwarz lemma.

(a) In fact, if $\mathcal{C} = \emptyset$, then $\mathcal{F}_{\mathcal{C}} = H^\infty$, so all bounded analytic functions are competing functions, and Theorem 1.1 is just the statement of Schwarz' lemma, which implies that $B_\emptyset$ is the identity map. In particular, the maximal Blaschke products without critical points, i.e., the locally univalent maximal Blaschke products are precisely the unit disk automorphisms.

(b) If $\mathcal{C} \neq \emptyset$ is a finite sequence and $N = 0$, then Theorem 1.1 is exactly Nehari's 1947 generalization of Schwarz' lemma (see [31], Corollary[1] to Theorem 1). In particular, if $\mathcal{C} = (z_1, \ldots, z_m)$ is a finite sequence consisting of m points, then every maximal Blaschke product with critical set $\mathcal{C}$ is a finite Blaschke product of degree $m+1$. As we shall see in Remark 3.2, the converse is also true. Hence the maximal Blaschke products with finitely many critical points are precisely the finite Blaschke products.

[1] In his formulation of this Corollary, Nehari apparently assumes, implicitly, that the origin is not a critical point. Otherwise, Nehari's statement concerning the case of equality would not be entirely correct.