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Surfaces of constant curvature in $\mathbb{R}^3$ with isolated singularities

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Abstract

We prove that finite area isolated singularities of surfaces with constant positive curvature $K > 0$ in $\mathbb{R}^3$ are removable singularities, branch points or immersed conical singularities. We describe the space of immersed conical singularities of such surfaces in terms of the class of real analytic closed locally strictly convex curves in $\mathbb{S}^2$ with admissible cusp singularities, characterizing when the singularity is actually embedded. In the global setting, we describe the space of *peaked spheres* in $\mathbb{R}^3$, i.e. compact convex surfaces of constant curvature $K > 0$ with a finite number of singularities, and give applications to harmonic maps and constant mean curvature surfaces.

Keywords: Surfaces of constant curvature; Isolated singularities; Conical singularities; Monge–Ampère equation; Harmonic maps; Geometric Cauchy problem; Peaked sphere

1. Introduction

It is a classical result that any complete surface of constant curvature $K > 0$ in $\mathbb{R}^3$ is a round sphere. Thus, if one extends by analytic continuation a local piece of such a *K-surface* in $\mathbb{R}^3$ other

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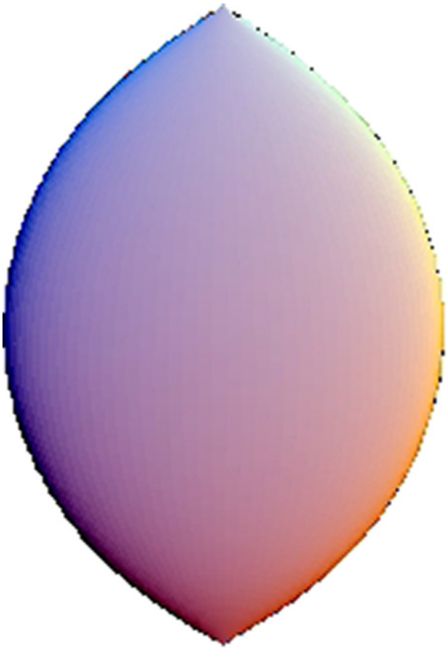

Fig. 1. A rotational peaked sphere ($K = 1$) in $\mathbb{R}^3$.

than a sphere, singularities will eventually appear. It is hence natural to consider K-surfaces in $\mathbb{R}^3$ in the presence of singularities, and to investigate how the nature of these singularities determines the global geometry of the surface.

Surfaces of positive constant curvature with singularities are related to other natural geometric theories. For instance, they are parallel surfaces to constant mean curvature surfaces, their Gauss map is harmonic into $\mathbb{S}^2$ for the conformal structure of the second fundamental form, and this Gauss map can often be realized as the vertical projection of a minimal surface in the product space $\mathbb{S}^2 \times \mathbb{R}$. Also, when viewed as graphs, these surfaces are solutions to one of the most widely studied elliptic Monge–Ampère equations:

$$u_{xx}u_{yy} - u_{xy}^2 = K(1 + u_x^2 + u_y^2)^2, \quad K > 0. \tag{1.1}$$

Thus, the regularity theory for K-surfaces in $\mathbb{R}^3$ is tightly linked to the regularity theory of Monge–Ampère equations.

It is remarkable that surfaces of positive constant curvature in $\mathbb{R}^3$ can be regularly embedded around an isolated singularity, as shown by the rotational example in Fig. 1. This type of singularities does not appear in many other geometric theories, and even for the case of K-surfaces in $\mathbb{R}^3$ they only appear in special circumstances. Still, K-surfaces in $\mathbb{R}^3$ having only isolated singularities are of central interest to the theory from several points of view. We explain this next, together with our contributions to the topic.

1. *Singularity theory.* The singularities of K-surfaces in $\mathbb{R}^3$ generically form curves in $\mathbb{R}^3$. Thus, K-surfaces only with isolated singularities constitute a rare phenomenon that happens when the surface with singularities has the biggest possible regularity (it is everywhere analytic except for some isolated points).