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Spectral property of Cantor measures with consecutive digits[☆]

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Abstract

We consider equally-weighted Cantor measures $\mu_{q,b}$ arising from iterated function systems of the form $b^{-1}(x+i)$, $i = 0, 1, \ldots, q-1$, where $q < b$. We classify the (q, b) so that they have infinitely many mutually orthogonal exponentials in $L^2(\mu_{q,b})$. In particular, if q divides b, the measures have a complete orthogonal exponential system and hence spectral measures. Improving the construction by Dutkay et al. (2009) [3], we characterize all the maximal orthogonal sets Λ when q divides b via a maximal mapping on the q-adic tree in which all elements in Λ are represented uniquely in finite b-adic expansions and we can separate the maximal orthogonal sets into two types: regular and irregular sets. For a regular maximal orthogonal set, we show that its completeness in $L^2(\mu_{q,b})$ is crucially determined by the certain growth rate of non-zero digits in the tail of the b-adic expansions of the elements. Furthermore, we exhibit complete orthogonal exponentials with zero Beurling dimensions. These examples show that the technical condition in Theorem 3.5 of Dutkay et al. (2011) [4] cannot be removed. For an irregular maximal orthogonal set, we show that under some conditions, its completeness is equivalent to that of the corresponding regularized mapping.

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Contents

1. Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. We say that μ is a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ so that $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ. If $\chi_\Omega dx$ is a spectral measure, then we say that Ω is a *spectral set*. The study of spectral measures was first initiated by B. Fuglede in 1974 [7], when he considered a functional analytic problem of extending some commuting partial differential operators to some dense subspace of L^2 functions. In his first attempt, Fuglede proved that any fundamental domains given by a discrete lattice are spectral sets with its dual lattice as its spectrum. On the other hand, he also proved that triangles and circles on $\mathbb{R}^2$ are not spectral sets, while some examples (e.g. $[0, 1] \cup [2, 3]$) that are not fundamental domains can still be spectral. From the examples and the relation between Fourier series and translation operators, he proposed a reasonable conjecture on spectral sets: *$\Omega \subset \mathbb{R}^s$ is a spectral set if and only if Ω is a translational tile*. This conjecture baffled experts for 30 years until 2004, Tao [23] gave the first counterexample on $\mathbb{R}^d$, $d \geq 5$. The examples were modified later so that the conjecture is false in both directions on $\mathbb{R}^d$, $d \geq 3$ [14,13]. It remains open in dimensions 1 and 2. Despite the counterexamples, the exact relationship between spectral measures and tiling is still mysterious.

The problem of spectral measures is as exciting when we consider fractal measures. Jorgensen and Pedersen [12] showed that the standard Cantor measures are spectral measures if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectral self-similar/self-affine measures were also found ([16,5] et al.). The construction of these spectral self-similar measures is based on the existence of the *compatible pairs (known also as Hadamard triples)*. It is still unknown whether all such spectral measures are obtained from compatible pairs. Having an exponential basis, the series convergence problem was also studied by Strichartz. It is surprising that the ordinary Fourier series of continuous functions converge uniformly for standard Cantor measures [22]. By now there are considerable amount of papers studying spectral measures and other generalized types of Fourier expansions like the Fourier frames and Riesz bases (see [2,4,6,8,10,11,15,16,18,20,21], and the references therein).

In [9], Hu and Lau made a start in studying the spectral properties of Bernoulli convolutions, the simplest class of self-similar measures. They classified the contraction ratios with infinitely