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Solving the Loewner PDE in complete hyperbolic starlike domains of $\mathbb{C}^N$

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Abstract

We prove that any Loewner PDE in a complete hyperbolic starlike domain of $\mathbb{C}^N$ (in particular in bounded convex domains) admits an essentially unique univalent solution with values in $\mathbb{C}^N$.

Keywords: Loewner theory; Holomorphic PDE; Evolution equations

1. Introduction

The classical Loewner theory in the unit disc of $\mathbb{C}$ was introduced by C. Loewner [22] in 1923 and later developed by P.P. Kufarev [21] and C. Pommerenke [25]. We refer the reader to [1] for a recent survey on applications and generalizations of such a theory.

In higher dimensions, J. Pfaltzgraff [23,24] extended the basic theory to $\mathbb{C}^N$, and later on many authors contributed to study the higher (or even infinite) dimensional Loewner ODE and PDE. Just to name a few, we recall here the contributions of T. Poreda [26], I. Graham, H. Hamada and G. Kohr [18,19].

More recently, the second named author together with M.D. Contreras and S. Díaz-Madrigal [8,9] generalized and solved the Loewner ODE on complete hyperbolic manifolds, and later the

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first two authors together with H. Hamada and G. Kohr [7] showed that the Loewner PDE on complete hyperbolic manifolds always admits an (essentially unique) abstract univalent solution with values in a complex manifold. The main remaining open problem, whether any Loewner PDE in a complete hyperbolic domain in $\mathbb{C}^N$ admits a univalent solution with values in $\mathbb{C}^N$, has been given many partial positive answers, see [14,18,19,3–5,27,20].

In the present paper we show that any Loewner PDE in a complete hyperbolic starlike domain in $\mathbb{C}^N$ (in particular in bounded convex domains and in the unit ball) admits a univalent solution with values in $\mathbb{C}^N$. For $N = 1$ it is known [10] that any Loewner PDE in the unit disc admits a univalent solution with values in $\mathbb{C}$, so in what follows we will focus on the case $N \geq 2$.

Referring the reader to Section 2 for definitions and comments, our main result can be stated in the following way:

Theorem 1.1. *Let $D \subset \mathbb{C}^N$ be a complete hyperbolic starlike domain. Let $G : D \times \mathbb{R}^+ \to \mathbb{C}^N$ be a Herglotz vector field of order $d \in [1, +\infty]$. Then there exists a family of univalent mappings $(f_t : D \to \mathbb{C}^N)$ of order d which solves the Loewner PDE*

$$\frac{\partial f_t}{\partial t}(z) = -df_t(z)G(z,t), \quad a.a.\ t \geq 0, \forall z \in D. \tag{1.1}$$

Moreover, $R := \cup_{t\geq 0} f_t(D)$ is a Runge and Stein domain in $\mathbb{C}^N$ and any other solution to (1.1) is of the form $(\Phi \circ f_t)$ for a suitable holomorphic map $\Phi : R \to \mathbb{C}^N$.

2. Generalized Loewner theory

In what follows we denote by $\mathbb{R}^+$ the semigroup of non-negative real numbers and by $\mathbb{N}$ the semigroup of non-negative integer numbers. Let $D \subset \mathbb{C}^N$ be a domain. Recall that a holomorphic vector field H on D is *semicomplete* if the Cauchy problem $\dot{x}(t) = H(x(t))$, $x(0) = z_0$ has a solution defined for all $t \in [0, +\infty)$ for all $z_0 \in D$. Semicomplete holomorphic vector fields on complete hyperbolic manifolds have been characterized in terms of the Kobayashi distance (see, *e.g.*, [6]).

Definition 2.1. Let $D \subset \mathbb{C}^N$ be a domain. A *Herglotz vector field of order $d \in [1, +\infty]$* on D is a mapping $G : D \times \mathbb{R}^+ \to \mathbb{C}^N$ with the following properties:

(i) The mapping $G(z, \cdot)$ is measurable on $\mathbb{R}^+$ for all $z \in D$.
(ii) The mapping $G(\cdot, t)$ is a holomorphic vector field on D for all $t \in \mathbb{R}^+$.
(iii) For any compact set $K \subset D$ and all $T > 0$, there exists a function $C_{T,K} \in L^d([0, T], \mathbb{R}^+)$ such that

$$\|G(z,t)\| \leq C_{T,K}(t), \quad z \in K, \text{a.a. } t \in [0, T].$$

(iv) $D \ni z \mapsto G(z, t)$ is semicomplete for almost all $t \in [0, +\infty)$.

Herglotz vector fields are strictly related to evolution families:

Definition 2.2. Let $D \subset \mathbb{C}^N$ be a domain. A family $(\varphi_{s,t})_{0\leq s\leq t}$ of holomorphic self-mappings of D is an *evolution family of order $d \in [1, +\infty]$* if it satisfies

$$\varphi_{s,s} = \mathsf{id}, \qquad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \quad 0 \leq s \leq u \leq t, \tag{2.1}$$