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# The Morse–Sard theorem for Clarke critical values

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**Abstract**

The Morse–Sard theorem states that the set of critical values of a $C^k$ smooth function defined on a Euclidean space $\mathbb{R}^d$ has Lebesgue measure zero, provided $k \geq d$. This result is hereby extended for (generalized) critical values of continuous selections over a compactly indexed countable family of $C^k$ functions: it is shown that these functions are Lipschitz continuous and the set of their Clarke critical values is null.



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## 1. Introduction

The classical Morse–Sard theorem states that the set of critical values of a $C^k$ smooth function defined on a Euclidean space $\mathbb{R}^d$ has Lebesgue measure zero, provided $k \geq d$ (see [11,16]). The result is sharp as far as the order of smoothness of the function is concerned [17,12]

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and is naturally extended to $C^k$-functions defined on a $C^k$-manifold of dimension $d$ (see, *e.g.*, [8, Theorem 1.3]). Quantitative Sard-type theorems are obtained in [5,18].

Generalized Morse–Sard results are known in variational analysis, under a generalized notion of criticality, usually defined in terms of the Clarke subdifferential [4] (the definition is recalled in Section 2). Positive results are known in some particular cases: we quote for instance [13] for the distance function to the Riemannian submanifold and [14] for viscosity solutions of Hamiltonians of certain type. The Morse–Sard theorem obviously fails for general Lipschitz functions – it already fails for $C^1$ functions in $\mathbb{R}^d$ with $d \geq 2$ – but the failure is of a different type: in the classical case, the failure of the Morse–Sard theorem is due to the (bad) structure of the set of critical points (smooth functions are constant on rectifiable arcs made up of critical points, as a consequence of the chain rule). On the contrary, in [3] it is shown that for a generic (for the uniform topology) set of 1-Lipschitz functions, the Clarke subdifferential at any point equals the ball $B(0, 1)$, that is, all points of a generic 1-Lipschitz function are critical (the chain rule fails generically for the Clarke subdifferential).

In recent years, tame variational analysis is an alternative way to circumvent smoothness and to deal with generalized critical values. This leads to considering particular subclasses of Lipschitz functions enjoying *a prior* structural assumption: Lipschitz (nonsmooth) functions whose graphs are definable in some $o$-minimal structure (see [6] for the relevant definition). Indeed, under this tame assumption, the Morse–Sard result follows as a consequence of the existence of a sufficiently smooth Whitney (normally regular) stratification, and the so-called projection formula for the Clarke subdifferential to the tangent space of the corresponding stratum [2, Proposition 4]. This leads to two fundamental results in tame variational analysis:

- Clarke critical values for tame Lipschitz functions are locally finite [2];
- tame closed graph set-valued maps, or more generally, set-valued maps admitting a sufficiently smooth normally regular stratification, satisfy the Sard theorem for the set of their critical values (values where metric regularity fails) [9].

We point out that $o$-minimality (tameness) is a strong structural condition (usually verified by means of the Tarski–Seidenberg principle) that restricts the universe of interest to a favorable subclass of objects (functions, operators). The above result shows that for tame nonsmooth functions the Morse–Sard theorem holds and its conclusion is reinforced (local finiteness instead of null-measure). Nevertheless, quite surprisingly, tameness does not automatically guarantee a Morse–Sard type result *for any* variational notion of criticality. For instance, let us call a point $\bar{x} \in \mathbb{R}^d$ broadly critical for the function $f$, if for every $\varepsilon > 0$, the closed convex hull of all derivatives $\nabla f(x)$ at points $x \in B(\bar{x}, \varepsilon)$ contains 0 (definability of the function guarantees differentiability in an open dense set). In [1], an example of a continuous (globally) subanalytic function $f : \mathbb{R}^3 \to \mathbb{R}$ which is strictly increasing in a segment of broadly critical points is presented, showing that the Morse–Sard theorem fails for this notion of criticality.

In contrast to the aforementioned tame-geometrical results, in this work we do not make any prior structural assumption on the nature of the functions. Instead, we consider the class of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ of the form

$$f(x) \in \{F(x, t) : t \in T\}, \quad \text{for all } x \in \mathbb{R}^d \tag{1}$$

where

($\mathcal{H}_1$) $T$ is a nonempty compact countable set;
($\mathcal{H}_2$) for each $t \in T$, the function $x \mapsto F(x, t)$ is $C^k$-smooth, with $k \geq d$;
($\mathcal{H}_3$) the functions $F$ and $\nabla_x F$ (defined on $\mathbb{R}^d \times T$) are continuous.