Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 239 (2013) 97–108

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Multivariate diagonal coinvariant spaces for complex reflection groups

François Bergeron

Département de Mathématiques, Université du Québec à Montréal, P.O. Box 8888, Succ. Centre-Ville, Montréal, H3S 3P8, Canada

Received 22 May 2011; accepted 26 February 2013
Available online 22 March 2013

Communicated by Adriano Garsia

Abstract

For finite complex reflection groups, we consider the graded W-modules of diagonally harmonic polynomials in r sets of variables, and show that associated Hilbert series may be described in a global manner, independent of the value of r.

Keywords: Diagonal coinvariant space; Diagonal harmonic polynomials; Multivariate; Finite complex reflection groups

Contents

1. Introduction 98
2. Definitions and discussion 98
   2.1. Diagonal coinvariant space 98
   2.2. Harmonic polynomials 99
   2.3. Main theorems 100
3. Proofs 105
4. Low degree components 106
5. Concluding remarks and thanks 107
   References 107

E-mail address: bergeron.francois@uqam.ca.

http://dx.doi.org/10.1016/j.aim.2013.02.013

1. Introduction

For finite complex reflection groups $W = G(m, p, n)$, we study the diagonal coinvariant space $\mathcal{C}_W$ for W, in several (say r) sets of n variables. Here, the use of the term diagonal refers to the fact that W is considered as a diagonal subgroup of W^r, acting on the rth-tensor power $\mathcal{R}_n^{(r)}$ of the symmetric algebra of the defining representation of W. The space considered, $\mathcal{C}_W^{(r)}$, is simply the quotient of $\mathcal{R}_n^{(r)}$ by the ideal generated by constant-term-free (diagonal) W-invariants. We shall see that the associated multigraded Hilbert series, denoted $\mathcal{C}_W^{(r)}(q_1, \ldots, q_r)$ (which is symmetric in the variables $\mathbf{q} := (q_1, \ldots, q_r)$, can be described in a uniform manner as a positive coefficient linear combination of Schur polynomials

$$\mathcal{C}_W^{(r)}(\mathbf{q}) = \sum_\mu c_\mu \, s_\mu(\mathbf{q}), \tag{1}$$

with the c_μ independent of r, and μ running through a finite set of integer partitions that depend only on the group W. This expression has the striking feature that it gives one global formula that specializes to the dimension of $\mathcal{C}_W^{(r)}$, for each individual r. To better see what is striking here, it is worth recalling that, although the case $r = 1$ has a long history [4,13,14], it is only recently that the special case $r = 2$ has been somewhat settled [3,6,7,9]. However much still needs to be done along these lines as discussed in [8]. Some headway has recently been made in the case $r = 3$ (see [2,11]), but the general case is still wide open.

2. Definitions and discussion

For a rank n complex reflection group W, we may consider its "diagonal" action on the $\mathbb{N}^r$-graded space $\mathcal{R}_n^{(r)} := \mathbb{C}[X]$, of polynomials in the $r \times n$ variables

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \cdots & x_{rn} \end{pmatrix}.$$

We consider each row of X as a set of n variables. Thinking of elements $w \in W$ as $n \times n$ matrices, the action of w is simply the map sending $f(X)$ to $f(X\,w)$. Naturally, *W-invariant polynomials* in $\mathbb{C}[X]$ are those that are fixed by any element of W, i.e.:

$$f(X \cdot w) = f(X), \quad \text{for all } w \in W.$$

2.1. Diagonal coinvariant space

The *diagonal coinvariant space* $\mathcal{C}_W^{(r)}$ is defined to be the quotient

$$\mathcal{C}_W^{(r)} := \mathcal{R}_n^{(r)} / \mathcal{I}_W^{(r)}, \tag{2}$$

where $\mathcal{I}_W^{(r)}$ is the ideal generated by constant-term-free W-invariant polynomials in $\mathcal{R}_n^{(r)}$.

For an integer $r \times n$ matrix $A = (a_{ij})$, we denote by X^A the monomial

$$X^A := X_1^{A_1} X_2^{A_2} \cdots X_n^{A_n}, \tag{3}$$